Discussion Paper No. B–395

Factor Models and the Shape of the Term Structure

Erik Schlögl and Daniel Sommer
University of Bonn, Department of Statistics,
Adenauerallee 24–42,
53113 Bonn, Germany

January 1997

JEL subject category number: G13

Keywords: multifactor term structure models, spread options, term structure shapes, forward rate curves, mean reversion

E-Mail: `schloegl@addi.finasto.uni-bonn.de`
`sommer@addi.finasto.uni-bonn.de`

Financial support by Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 303 at the University of Bonn, is gratefully acknowledged.

FACTOR MODELS AND THE SHAPE OF THE TERM STRUCTURE

ERIK SCHLÖGL AND DANIEL SOMMER

Abstract. The present paper analyses a broad range of one– and multifactor models of the term structure of interest rates. We assess the influence of the number of factors, mean reversion, and the factor probability distributions on the term structure shapes the models generate, and use spread options as an aggregate measure of the relative importance assigned to rising and falling forward rate curves by the models considered. We derive valuation formulas for these contingent claims in the multifactor Gaussian and CIR-models. Our main result is that the specification of mean reversion and the number of factors are both much more important for the relative movements of interest rates than the distributional characteristics of the factors. To the extent that interest rate risk depends on the movements of different parts of the term structure relative to one another rather than on shifts of its absolute level, the distributional assumption on the factor dynamics is found to be essentially irrelevant.

Introduction

For the valuation of fixed income derivative securities a large number of models have been suggested in the literature, both in discrete and continuous time. Of these, factor models represent an important subclass, which will be the object of our analysis. One–factor models include the continuous time limit of the Ho and Lee (1986) model and the Vasicek (1977) model, where continuously compounded rates are normally distributed, the Cox, Ingersoll jr. and Ross (1985) (CIR) model, where the short rate obeys a noncentral chi–square distribution, and the Brennan and Schwartz (1977) and Black and Karasinski (1991) models with log–normal short rates.

In multifactor models for reasons of analytical tractability we concentrate on the class of affine term structure models of the type analysed by Duffie and Kan (1992, 1996). More specifically we look at multifactor Gaussian[1] and CIR models, which are the most intensively studied[2] in the literature.

There are a number of empirical studies that try to single out the most suitable from a given set of term structure models suggested in the theoretical literature. For one–factor models, Chan, Karolyi, Longstaff and Sanders (1992) (CKLS) provide one answer in estimating the parameters of a general diffusion process on the basis of short rate data. However, if these models purport to describe the arbitrage–free dynamics of the entire term structure of interest rates, then they should be tested under this premise, i.e. whether they can explain observed term structures. This is the approach taken by Stambaugh (1988), Chen and Scott (1993) and Brown and Schaefer (1994), among others. Since the pricing of fixed income derivatives is the aim of term structure modelling, empirical studies have been conducted comparing observed prices of derivatives with model prices, for example by Flesaker (1993), who tests the Ho/Lee model. Given the normative nature of arbitrage pricing models, one might argue that a more convincing test of model prices is whether one

Date. January 31, 1997.

Financial support by Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 303 at the University of Bonn, is gratefully acknowledged.

[1]Gaussian in the sense that continuously compounded rates are normally distributed.

[2]El Karoui, Lepage, Myneni, Roseau and Viswanathan (1991) provide a comprehensive exploration of Gaussian models. For multifactor CIR models see for example Chen and Scott (1995).

can take advantage of deviations from observed market prices to acquire wealth. Cohen and Heath (1992) and Amin and Morton (1994) analyse term structure models under this aspect. However, depending on the approach the results of the empirical literature are fairly inconclusive.

Our aim is to contribute to the discussion on which term structure model is best suited for the pricing and hedging of interest rate sensitive derivative instruments by clarifying the implications that different specifications of the factor dynamics have for the evolution of the term structure. The question what type of changes of the shape of the term structure can be explained by the dynamics endogenous to the models has bearing on the effectiveness of the respective hedging strategies, because changes for which the models do not allow are not hedged. The factor models which we consider differ in three respects: the number of factors, the class of probability distributions that the factors obey, and whether the factor SDEs exhibit mean reversion or not. We identify the relative importance of these three choices for the endogenous evolution of the term structure[3]. Our main result is that the specification of mean reversion and the number of factors are both much more important for the relative movements of interest rates than the distributional characteristics of the factors. To the extent that interest rate risk depends on the movements of different parts of the term structure relative to one another rather than on shifts of its absolute level, the distributional assumption on the factor dynamics is found to be essentially irrelevant.

The rest of the paper is divided into three main sections. After introducing the models considered, we analyse the role of mean reversion and the importance of the number of factors for the term structure dynamics permitted. Here we observe strong qualitative similarities between models with different distributional assumptions, which we quantify in the subsequent section using values of European spread options on the forward rate curve as an aggregate measure of its shape. We conclude the paper with a summary of our results and some remarks on how they pertain to empirical testing.

1. Models

Consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. Assume that we are given an n–dimensional Wiener process, W, on this probability space and that $\{\mathcal{F}_t\}_{t\geq 0}$ is the augmented natural filtration of this Wiener process. In all the models considered the short rate, r, is assumed to follow an Itô process of the following type

$$dr(t) = \mu(t, z(t))dt + \sigma(t, z(t))dW(t). \tag{1}$$

Here z is a vector process of state variables that is assumed to be the unique strong solution to some vector stochastic differential equation (SDE) on the above probability space.

Define the savings account $(\beta_t)_{t\geq 0}$ by $\exp\left\{\int\limits_0^t r(s)ds\right\}$. We assume that r is sufficiently regular so that the price process of a zero coupon bond with maturity T, $B(\cdot, T)$, can be defined as a continuous version of the martingale

$$\big(B(t,T)\big)_{0\leq t\leq T} := \left(E^P\left[\exp\left\{-\int\limits_t^T r(s)ds\right\}\middle|\mathcal{F}_t\right]\right)_{0\leq t\leq T}.$$

[3]Rogers (1995) approaches the question "Which model for the term–structure of interest rates should one use?" differently, discussing a broader range of issues from analytical tractability to the compatibility of the models with the framework of equilibrium theory.

Hence we assume P to be the "risk–neutral" probability measure in the sense that the price process of every contingent claim that is attainable through a hedging strategy in zero coupon bonds is given by the conditional expectation under P of the payoff of this contingent claim discounted by the savings account.

Given price processes for zero coupon bonds we can define processes of instantaneous forward rates by

$$r_c(\cdot, T) := -\frac{\partial \ln B(\cdot, T)}{\partial T}. \tag{2}$$

Following Jamshidian (1987), we will sometimes have reason to employ the *forward risk adjusted measure*. For some time T this measure, Q^T, is defined by the following Radon/Nikodym derivative[4]

$$\frac{dQ^T}{dP} = \frac{B(T,T)}{B(0,T)\exp\left\{\int\limits_0^t r(s)ds\right\}}$$

1.1. **One–Factor Models.** A whole class of one–factor models is obtained by specifying (1) as (Hull and White (1993))

$$dr(t) = (\theta(t) - ar_t)dt + \sigma r(t)^\beta dW(t) \tag{3}$$

where $\theta(t)$ is a deterministic function of t and $a \geq 0$, $\sigma > 0$ and $\beta \geq 0$ are constants. This family includes the generalized[5] Vasicek (1977) model[6] for $\beta = 0$, Cox, Ingersoll jr. and Ross (1985) for $\beta = 0.5$, and Brennan and Schwartz (1977) for $\beta = 1$. Furthermore, in an empirical study of US Treasury bill yield data, Chan, Karolyi, Longstaff and Sanders (1992) (CKLS) estimate β as 1.5. Setting $\beta \in]0; 0.5[$ does not make sense, because in such a case the solution to (3) is not unique[7].

We complement our analysis of these specifications by looking at the Black and Karasinski (1991) and Sandmann and Sondermann (1993) models, whose short rate diffusions are

$$dr(t) = r(t) \cdot (\theta(t) - a \ln r(t) + \frac{1}{2}\sigma^2)dt + r(t)\sigma dW(t) \tag{4}$$

and

$$dr(t) = (1 - e^{-r(t)})[(\theta(t) - \frac{1}{2}(1 - e^{-r(t)})\sigma^2)dt + \sigma dW(t)] \tag{5}$$

respectively.

With these models we cover a wide array of distributional assumptions on the short rate suggested in the literature[8], ranging from continuously compounded rates which are

[4] For a rigorous treatment of changes of measure/changes of numéraire and their application to option pricing, see El Karoui and Rochet (1989) or Geman, El Karoui and Rochet (1995).

[5] We use the term "generalized" to indicate that the drift parameter $\theta(t)$ depends on time in such a manner as to fit the model to the initial term structure, as proposed in Hull and White (1990).

[6] Including the continuous time limit of the Ho and Lee (1986) model as a special case with $a = 0$.

[7] See for example Arnold (1973), p. 124.

[8] One can of course imagine further additions to the class of one–factor short rate models. See for example Schlögl and Sommer (1994).

normally distributed for $\beta = 0$, non-central chi–square distributed for $\beta = 0.5$ and lognormally distributed for $\beta = 1$ and in the Black and Karasinski (1991) model, to lognormally distributed nominal[9] short rates in the Sandmann and Sondermann (1993) model[10].

1.2. **Multifactor Models.** As the number of factors increases, numerical evaluation of the models becomes more and more difficult, and thus closed–form and near–closed–form solutions grow in importance. Therefore in the multifactor case we focus on Gaussian and CIR-type models. Since Gaussian models offer the greater analytical tractability and in order to make the models quantitatively comparable, we begin with a multifactor version of the CIR model and then construct a Gaussian model which fits the same initial term structure and allows us to match factor variances.

1.2.1. *A Multi-factor "Square-Root" Term Structure Model.* As in Chen and Scott (1995), the instantaneous risk-free interest rate (in the following: short rate) is assumed to be the sum of independent state variables, or factors

$$r(t) = \sum_{j=1}^{n} z_j(t). \tag{6}$$

The state variable dynamics are of the CIR type:

$$dz_j(t) = (\theta_j - a_j z_j(t))dt = \sigma_j \sqrt{z_j(t)} dW_j(t)$$

or, in the vector notation

$$dZ(t) = (\theta - AZ(t))dt + V \begin{pmatrix} \sqrt{z_1(t)} \\ \vdots \\ \sqrt{z_n(t)} \end{pmatrix} dW(t)$$

Note that A and V are diagonal matrices. In order to keep the model analytically tractable, we do not allow for time-dependent θ or V in this case. Therefore, it will not be possible to fit arbitrary initial term structures. However, since the aim of our analysis is the term structure movements endogenously generated by the model, this is not a serious restriction.

The CIR bond price formula for zero coupon bonds with maturity T at time t, generalized to the multifactor case, is given in Chen and Scott (1995):

$$B(Z(t), t, T) = \mathcal{A}(t, T) \exp\{-\mathcal{B}(t, T) Z(t)\} \tag{7}$$

with

$$\mathcal{A}(t, T) = \prod_{j=1}^{n} \mathcal{A}_j(t, T)$$

$$\mathcal{A}_j(t, T) = \left(2c_j w_j \exp\left\{\frac{1}{2}(c_j + a_j)(T - t)\right\}\right)^{\frac{2\theta_j}{\sigma_j^2}}$$

$\mathcal{B}(t, T)$ a row vector with components

$$\mathcal{B}_j(t, T) = 2w_j(\exp\{c_j(T - t)\} - 1)$$

[9]We use the term "nominal" to differentiate between continuously coumpounded rates and actuarial compounding, and not "nominal" as opposed to "real" interest rates.

[10]An important feature of this specification is that it avoids the problem of infinite expected roll–over returns encountered in the Black/Karasinski model or when setting $\beta \geq 0.5$ in the β–root process. See Sandmann and Sondermann (1993) and Hogan and Weintraub (1993).

and

$$\begin{aligned} w_j =& ((c_j + a_j)\exp\{c_j(T-t)\} + c_j - a_j)^{-1} \\ c_j =& \sqrt{a_j^2 + 2\sigma_j^2} \end{aligned}$$

Let

$$\begin{aligned} \tilde{z}_j(T) &:= z_j(T)\cdot 2(w_j\sigma_j^2(e^{c_j(T-t)}-1))^{-1} \\ &= \frac{4}{\sigma_j^2}\mathcal{B}_j(t,T)^{-1}z_j(T). \end{aligned} \tag{8}$$

Then, following Jamshidian (1987), we know that $\tilde{z}_j(T)$ conditioned on $z_j(t)$ is noncentral chi-square distributed under the forward risk adjusted measure Q^T, with ν_j degrees of freedom and non-centrality parameter λ_j given by

$$\begin{aligned} \nu_j &= \frac{4\theta_j}{\sigma_j^2} \\ \lambda_j &= \frac{16w_j^2c_j^2\exp\{c_j(T-t)\}}{\sigma_j^2\mathcal{B}_j(t,T)}z_j(t) \end{aligned}$$

Furthermore, variances of the factors $\tilde{z}_j(T)$, viewed from time t under the measure Q^T, are given by[11]

$$\begin{aligned} &\mathrm{Var}[\tilde{z}_j(T)|\tilde{z}(t)] = 2(\nu_j + 2\lambda_j) \\ \Leftrightarrow\quad &\mathrm{Var}[z_j(T)|z(t)] = \sigma_j^2\mathcal{B}_j(t,T)\left(\frac{1}{2}\theta_j\mathcal{B}_j(t,T) + 4w_j^2c_j^2e^{c_j(T-t)}z_j(t)\right) \end{aligned} \tag{9}$$

1.2.2. *A comparable Gaussian model.* The short rate is again assumed to be driven by independent factors but now the state variable dynamics are of the generalized Vasicek (1977) type:

$$dz_j(t) = (\theta_j(t) - a_jz_j(t))dt + \sigma_j dW_j(t) \tag{10}$$

The one– and two–factor cases of this model are analysed in El Karoui, Lepage, Myneni, Roseau and Viswanathan (1991); the n–factor case can be treated analogously. In order to make this model comparable to the "square root" model introduced in the previous section, we will do the following: First, we set the mean reversion coefficients a_j equal to the respective coefficients in the "square root" model. Second, we choose the volatility parameters σ_j in such a manner as to keep the state variable variances for a specific time horizon equal in the "square root" and Gaussian models. Third, the time dependent drift coefficients $\theta_j(t)$ allow us to fit the initial term structure as demonstrated for the one–factor case in Hull and White (1990), where we take the (endogenous) initial term structure from the "square root"model as input for the Gaussian case. In this model logarithmic zero coupon bond prices are given by

$$\begin{aligned} \ln B(Z(t),t,T) = \ln\frac{B(Z(0),0,T)}{B(Z(0),0,t)} &- \frac{1}{2}\sum_{j=1}^{n}\int_0^t \frac{\sigma_j^2}{a_j^2}\left(\left(1-e^{-a_j(T-s)}\right)^2 - \left(1-e^{-a_j(t-s)}\right)^2\right)ds \\ &+ \sum_{j=1}^{n}\int_0^t \frac{\sigma_j}{a_j}\left(e^{-a_j(T-s)} - e^{-a_j(t-s)}\right)dW_j(s) \end{aligned} \tag{11}$$

[11]For the variance of chi-square distributed variables, see Johnson and Kotz (1970), p. 134.

Note that the dependence on $\theta_j(t)$ is subsumed in the logarithm of the initial bond prices with the respective maturities. Changing to the time T^* forward measure Q^*, where

$$dW_j^{Q^*}(t) = dW_j(t) + \frac{\sigma_j}{a_j}\left(1 - e^{-a_j(T^*-t)}\right) dt$$

are independent standard Wiener processes, we can write (11) as

$$\begin{aligned}
\ln B(Z(t), t, T) = \ln \frac{B(Z(0), 0, T)}{B(Z(0), 0, t)} &- \frac{1}{2}\sum_{j=1}^{n} \int_0^T \frac{\sigma_j^2}{a_j^2}\left(\left(1 - e^{-a_j(T-s)}\right)^2 - \left(1 - e^{-a_j(t-s)}\right)^2\right) ds \\
&- \sum_{j=1}^{n} \int_0^T \frac{\sigma_j^2}{a_j^2}\left(1 - e^{-a_j(T^*-s)}\right)\left(e^{-a_j(T-s)} - e^{-a_j(t-s)}\right) ds \\
&+ \sum_{j=1}^{n} \int_0^t \frac{\sigma_j}{a_j}\left(e^{-a_j(T-s)} - e^{-a_j(t-s)}\right) dW_j^{Q^*}(s) \qquad (12)
\end{aligned}$$

Variances of state variables are the same under the risk–neutral and forward risk adjusted measures, and are given by

$$\operatorname{Var}[z_j(t)] = \int_0^t e^{-2a_j(t-s)}\sigma_j^2 ds = \frac{\sigma_j^2}{2a_j}\left(1 - e^{-2a_j t}\right) .$$

We choose σ_j in the Gaussian model such that

$$\operatorname{Var}[z_j(t)] = \operatorname{Var}_{\chi^2}[z_j(t)]$$

Thus

$$\sigma_j^2 = 2a_j\left(1 - e^{-2a_j t}\right)^{-1} \operatorname{Var}_{\chi^2}[z_j(t)]$$

where $\operatorname{Var}_{\chi^2}[z_j(t)]$ is given by (9).

2. Term Structure Shapes

2.1. **Affine Models.** Setting $n = 1$ in section 1.2.2, we get the generalized Vasicek (1977) model. Then instantaneous forward rates (see eq. (2)) can be wriiten in terms of the short rate as

$$r_c(t, T) = r_c(0, T) + \frac{\sigma^2}{2a^2}(1 - e^{-2at})(e^{-a(T-t)} - e^{-2a(T-t)}) + e^{-a(T-t)}(r(t) - r_c(0, t)) \qquad (13)$$

and we can state

Proposition 1. *Consider a flat initial forward rate curve, i.e. instantaneous forward rates are equal for all maturities. Then the following holds:*

1. *Forward rate curves in the Ho/Lee model are increasing. Their slope is deterministic.*
2. *Forward rate curves in the Vasicek model are of three types*
   (a) *monotonically increasing*
   (b) *monotonically decreasing*
   (c) *humpshaped, that is they posses an interior maximum.*

Proof:

1. This follows immediately when letting $a \to 0$ in (13) and differentiating with respect to $\tau := T - t$.

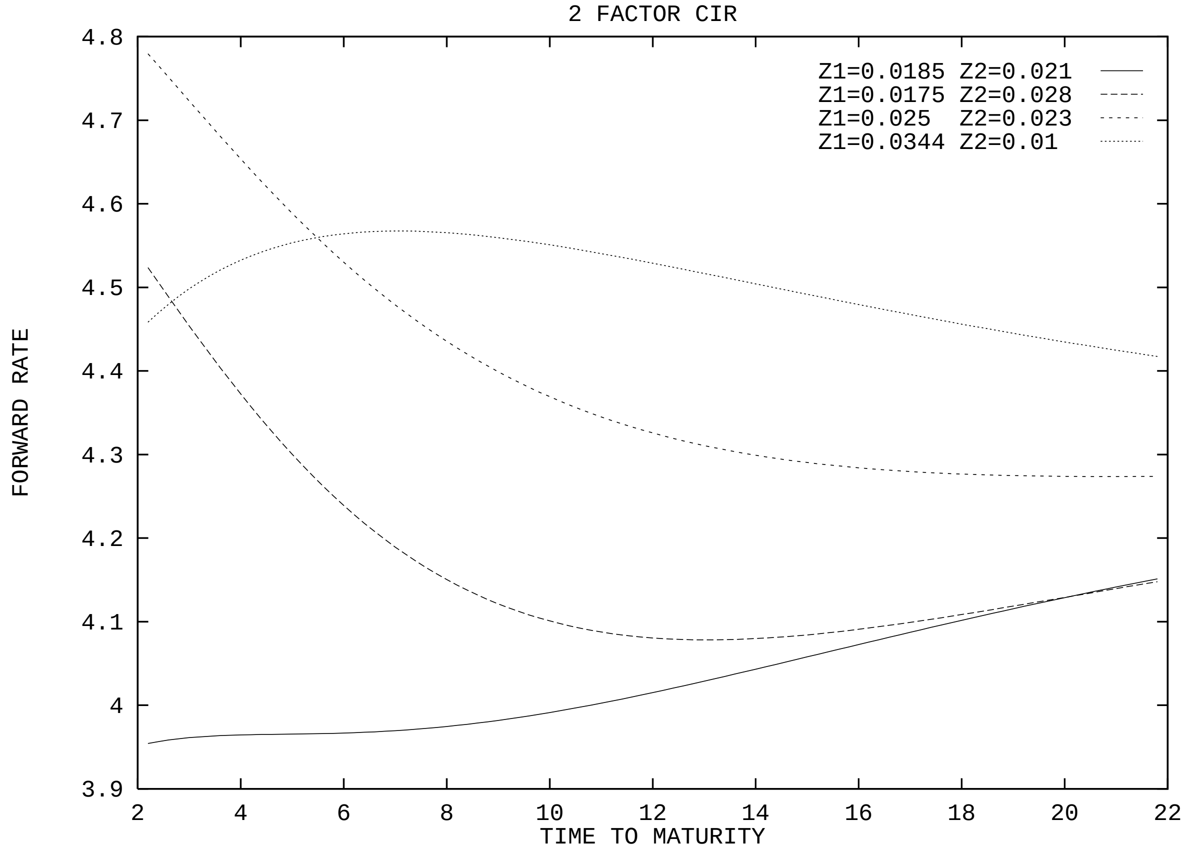

FIGURE 1. Term structure realizations, two–factor CIR model

2. From equation (13) the slope of the forward rate curve is

(14)

$$S_t(\tau) := \frac{\partial r_c(t,T)}{\partial \tau} = \frac{\partial r_c(0,T)}{\partial \tau} + \underbrace{e^{-a(T-t)}}_{>0}\left(a(r_c(0,t) - r(t)) + \underbrace{\frac{\sigma^2}{2a}(1 - e^{-2at})}_{>0}(2e^{-a(T-t)} - 1)\right)$$

As $2e^{-a(T-t)} - 1$ is monotonically decreasing in τ and bounded, the expression in brackets will either be positive or negative or have a unique zero depending on the value of $(r_c(0,t) - r(t))$. If it has a zero at $\overline{\tau}$, then $S_t(\tau) > 0$ for $\tau < \overline{\tau}$ and $S_t(\tau) < 0$ for $\tau > \overline{\tau}$. □

By starting with a flat initial forward rate curve these effects are most evident, but they carry over to scenarios with arbitrary initial term structures, since as can be easily seen in equation (13), the initial curve is simply added to the endogenously generated shape of the term structure at time t.

By restating the Cox/Ingersoll/Ross result on the shape of the yield curve[12] in terms of forward rates, we get the CIR analogue of 1:

Proposition 2. *Forward rate curves in the one–factor CIR model are of three types*

1. *monotonically increasing*
2. *monotonically decreasing*
3. *humpshaped, that is they posses an interior maximum.*

PROOF: See appendix A.

[12]see Cox, Ingersoll jr. and Ross (1985), p. 394

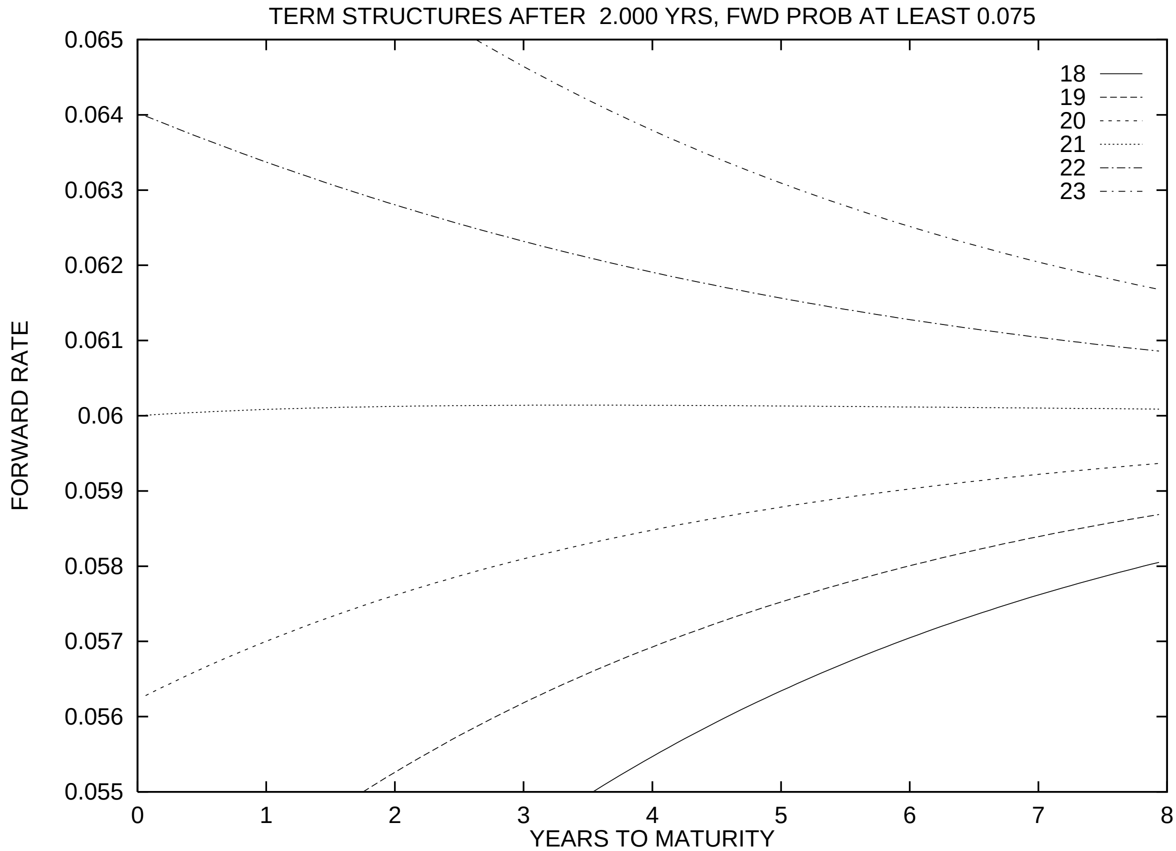

FIGURE 2. Brennan/Schwartz term structure realizations

The results on possible realizations of the term structure obtained for one–factor models in propositions 1 and 2 generalize to the multifactor case in a straightforward manner. Observe that the multifactor equation (11) is simply an affine combination of one–factor equations. More specifically, the term structure at some future date t in the Vasicek model is given by the the initial term structure at time zero (the logarithm of initial bond prices in (11) or the initial forward rate $r_c(0,T)$ in (13)) plus a term reflecting the endogenous dynamics. In the multifactor Gaussian model, we have again the initial term structure, plus a sum of endogenously generated terms of the same functional form for each factor. Although the formulas are more complicated, the same holds for the one– and multifactor versions of the CIR model, since as in the Gaussian case forward rates are affine functions of the state variables. Thus, starting from a flat initial term structure, an n–factor Gaussian or CIR model permits those term structure shapes which can be represented as a sum of n curves of the three types in propositions 1 and 2.

Moving from one–factor to multifactor models we do gain a new quality, however. The one–factor models allow only for "parallel" shifts of the term structure in the sense that forward rate curves for different states of the world cannot intersect (e.g. figure 2). A two–factor model in contrast allows for "twists" in the term structure, i.e. for example an increase of forward rates on the short end and a decrease on the long end (e.g. figure 1).

2.2. **Other Models.** Leaving the affine class for the remaining one–factor models which we want to study, we need to resort to numerical implementations. This also allows us to fit the CIR model to a given initial term structure by allowing the drift parameter θ to depend on time.

For our numerical study we chose the Hull and White (1993) algorithm, because it allows us to implement all one–factor models introduced in section 1.1 in a unified framework. Comparing the initial term structure with the period 0 term structure calculated by backward induction through the trinomial lattice generated by this algorithm gives us an idea of how exact our approximation is. The maximum deviations of the calculated period 0 term structure from the input term structure, as well as the parameter constellations for each plot, are listed in table D.

As exemplified in figure 2, our simulations show that for all values of β considered, as well as for the Black and Karasinski (1991) model, the shapes of the term structure endogenously generated are qualitatively the same as those proven for $\beta = 0, 0.5$ in propositions 1, 2. The Sandmann and Sondermann (1993) model exhibits essentially no mean reversion, therefore the term structure shapes generated endogenously remain very flat, with a slight tendency toward increasing forward rate curves.

2.3. **Some Remarks on Mean Reversion.** The analysis in the preceding sections demonstrates the importance of mean reversion for the shape of the term structure. Our numerical study shows that for all models considered, mean reversion is necessary in order to generate downward sloping term structures in any substantial proportion. As shown analytically in proposition 1, in the Gaussian case this means that for zero mean reversion (the continuous–time Ho/Lee case) the forward rate curve slopes upward ever more steeply as the model evolves over time. In the CIR model mean reversion is necessary in order to make the origin inaccessible. Otherwise, within any finite time horizon, the short rate would reach zero with a positive probability and stay there.[13]

In this context it is important to note that mean reversion is primarily a *volatility* parameter, in that it determines how the volatilities of bonds, and hence yields and forward rates of different maturities relate to the volatility of the short rate. Mean reversion is necessary in order to model a volatility structure which reflects the stylized fact that forward rate volatilities are decreasing in time to maturity.

An equivalent change of measure, much as it will alter the drift of the short rate SDE, will neither affect the volatility of the short rate nor the bond volatilities. Consequently, the term structures which are possible remain the same under every equivalent measure, only the probability assigned to the realizations change. In the affine models this is immediately evident: the term structure equations, for example (13) or (7), are independent of the probability measure.

In Gaussian models, choosing different rates of mean reversion for each factor is necessary to prevent collapse to the one–factor case: Then equation (11) can be written as

$$\ln B(Z(t),t,T) = \ln\frac{B(Z(0),0,T)}{B(Z(0),0,t)} - \frac{1}{2}\left(\sum_{j=1}^{n}\sigma_j^2\right)\int_0^t \frac{1}{a^2}\left(\left(1-e^{-a(T-s)}\right)^2 - \left(1-e^{-a(t-s)}\right)^2\right)ds$$

$$+\int_0^t \frac{1}{a}\left(e^{-a(T-s)} - e^{-a(t-s)}\right) d\left(\sum_{j=1}^{n}\sigma_j W_j(s)\right)$$

Defining

$$\tilde{W}(s) := \frac{\sum_{j=1}^{n}\sigma_j}{\sqrt{\sum_{j=1}^{n}\sigma_j^2}} W_j(s)$$

[13] see Cox, Ingersoll jr. and Ross (1985), p. 391.

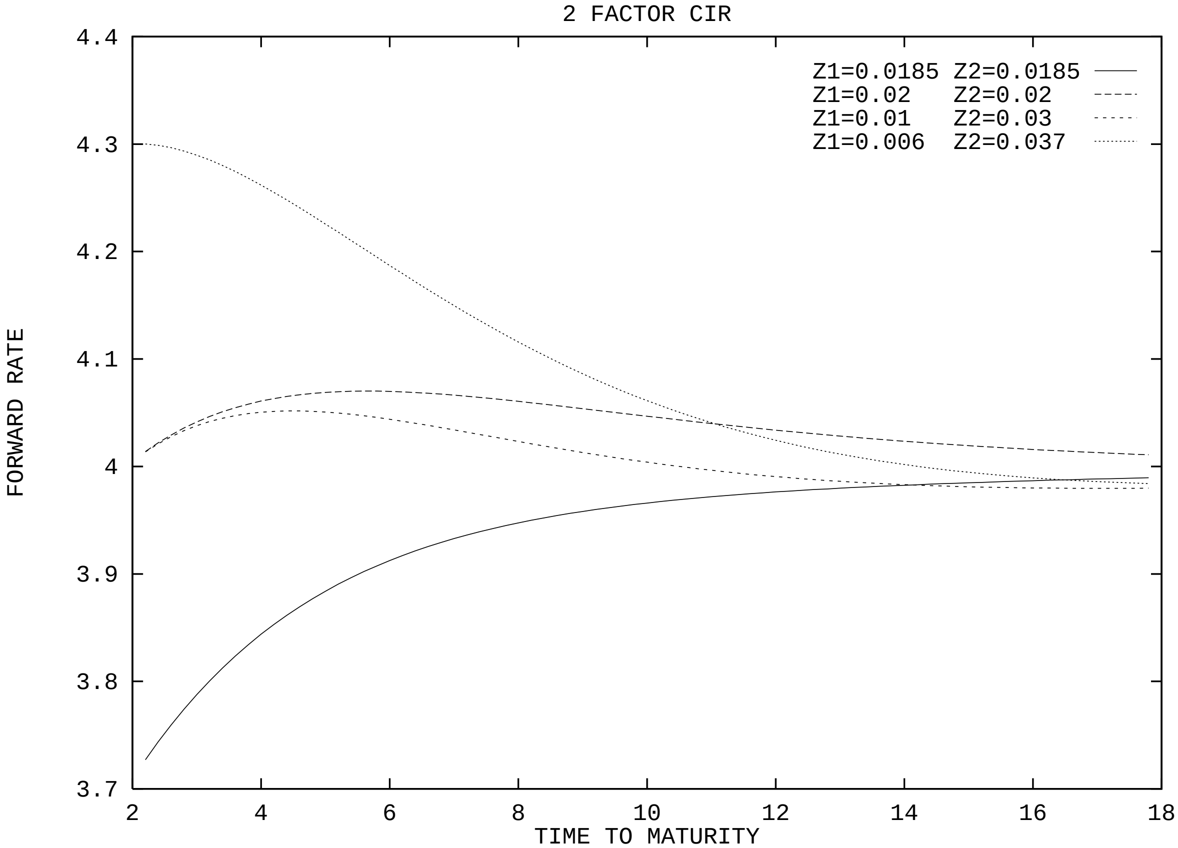

FIGURE 3. Term structure realizations, two–factor CIR model

which is a Wiener process by the Lévy criterion. Setting $\sigma := \sqrt{\sum_{j=1}^{n}\sigma_j^2}$, we can replace $d(\sum_{j=1}^{n}\sigma_j dW_j(s))$ with $\sigma d\tilde{W}(s)$ to yield a one–factor model. Consequently, it is not possible to construct a continuous–time multifactor version of the Ho/Lee model, since this would have $a_j = 0$ for all j.

In contrast, the multifactor version of the CIR model does not degenerate into the one–factor case if the mean reversion parameters are equal for all factors, as figure 3 demonstrates. Only if the volatility parameters σ_j are also all equal do we get a one–factor model. Then the $\mathcal{B}_j(t,T)$ are equal for all j and we can write (7) as

$$B(Z(t),t,T) = \mathcal{A}(t,T)\exp\{-\sum_{j=1}^{n}\mathcal{B}_j(t,T)z_j(t)\} = \mathcal{A}(t,T)\exp\{-\mathcal{B}_1(t,T)\,r(t)\}$$

3. The Role of the Distributional Assumption

In the previous section, our qualitative results on the shapes of the term structure generated endogenously were independent of the distributional characteristics of the factors. In this section we seek to quantify this similarity between the models. To this end consider a European spread option whose payoff at time T is defined as

$$[r_c(T,T+x) - r_c(T,T+y)]^+.$$

Such spread options can be interpreted as an aggregate over all possible term structure shapes, weighted with the relevant pricing measure: If (long – short)–options (i.e. $x > y$) are expensive, then increasing forward rate curves carry a large weight; if (short – long)–options (i.e. $x < y$) are expensive, then decreasing term structures are important. Figures 4–7 plot spread option values for different one–factor models of the "β–root" specification

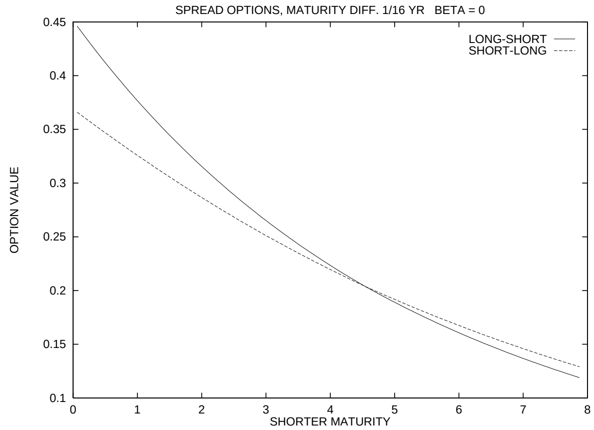

FIGURE 4. $\beta = 0$

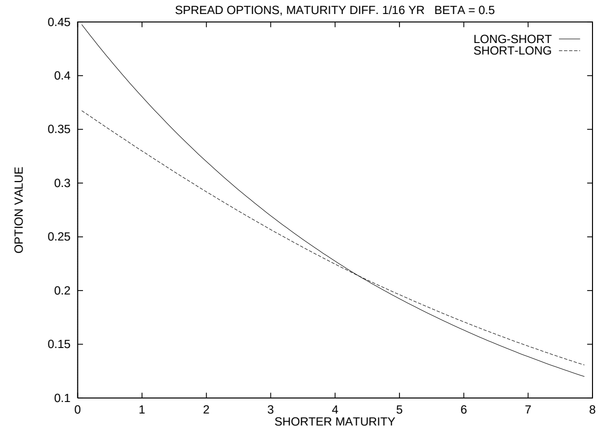

FIGURE 5. $\beta = 0.5$

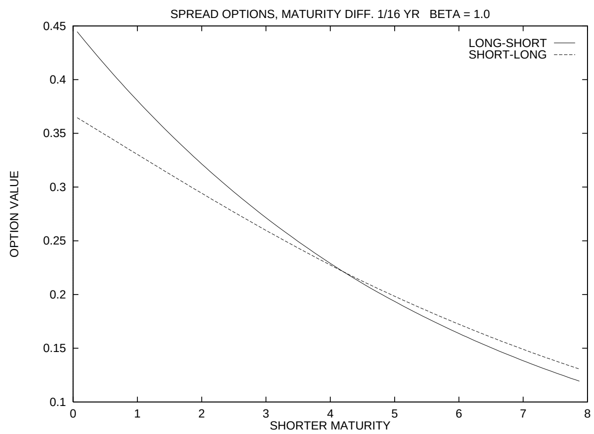

FIGURE 6. $\beta = 1$

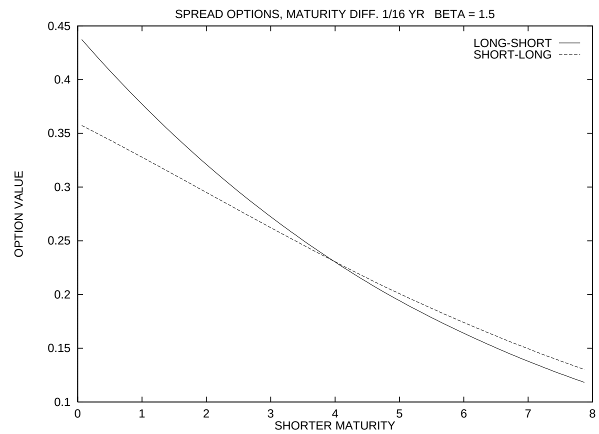

FIGURE 7. $\beta = 1.5$

(3). On the vertical axis we plot the price of a contingent claim which pays one dollar for every base point difference between the instantaneous forward rates with time to maturity x years hence and $x+c$ years hence, i.e. either long – short $[r_c(T, T+x+c) - r_c(T, T+x)]^+$, or short – long $[r_c(T, T+x) - r_c(T, T+x+c)]^+$. On the horizontal axis we plot the shorter time to maturity x, keeping the maturity difference c constant. We start with a flat initial forward rate curve and chose the volatility parameter σ in such a manner as to make the models comparable: Since the short rate dynamics drive the one–factor models considered here, we set σ so as to match the variance of the short rate at option maturity as viewed from today.[14] Figures 4–7 show that the models are not only very similar in the qualitative shapes of the term structure generated endogenously, but also differ very little quantitatively in they weight they assign to different slopes of the forward rate curve at various maturities.

It is worth noting that the endogenous dynamics identified so far are unaffected by the shape of the initial term structure. This was already observed analytically in section 2.1. In our numerical study, by defining "at–the–money" spread options, i.e. options that pay one dollar for every base point difference between the spread between two forward rates at maturity of the option and the respective spread in the initial term structure $[(r_c(0, T+x+d) - r_c(0, T+x)) - (r_c(T, T+x+c) - r_c(T, T+x))]^+$, we verified that all these models have a tendency to produce term structures that are more upward sloping

[14]This of course does not mean that the short rate variance is matched for other maturities, but our simulations show that the variance matching σ does not vary much across maturities. Furthermore, by allowing for time–dependent volatility one could match variances for all maturities.

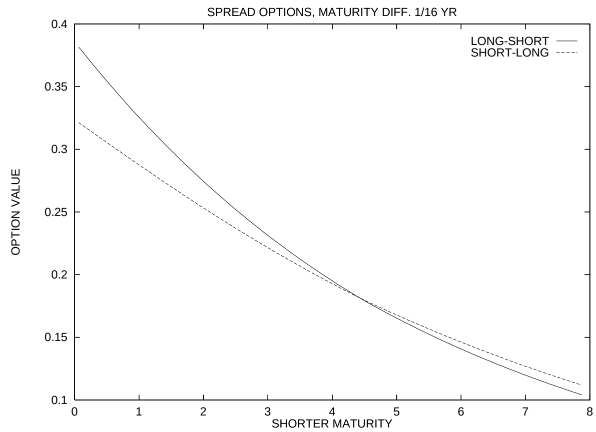

FIGURE 8. Black/Karasinski

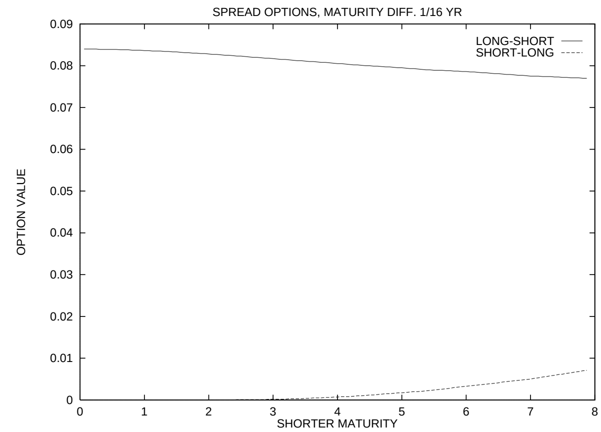

FIGURE 9. Sandmann/Sondermann

on the short and more downward sloping on the long end irrespective of the slope of the initial term structure.

We therefore conclude that by specifying different values of β, it is not possible to implement term structure models which differ substantially with respect to their structural implications for future realizations of the shape of the term structure.

The Black and Karasinski (1991) model differs from the "β–root" diffusion in the way mean reversion is specified (see eq. (4). This is reflected in figure 8, where again we set σ so that the variance of the short rate in two years' time is the same as in figures 4 through 7: The forward rate curves are somewhat flatter, leading to lower spread option prices.

In the Sandmann and Sondermann (1993) model the term that can be viewed as generating a mean reversion effect $\frac{1}{2}(1 - e^{-r(t)})\sigma^2$, is bounded between 0 and $\frac{1}{2}\sigma^2$ for all $r \in \mathbb{R}_+$. For the same variance of the short rate realizations in two years' time as in figure 4 the term structures are therefore flatter and upward sloping term structures carry relatively more weight, as evidenced in figure 9: Spread option prices are much lower and for all maturities the claims contingent on upward sloping term structures are more valuable than those contingent on downward sloping term structures. This underscores once more the importance of mean reversion as compared as compared to the distributional characteristics.

Next we pose the question whether these striking similarities between models with different distributional assumptions carry over to the multifactor case. Again we focus on members of the affine class. While we do not have closed form solutions for spread options on instantaneous forward rates in the CIR model, in appendix C we derive a near–explicit solution for spread options on nominal[15] forward rates[16].

Consider now figure 10. The CIR θ_j were chosen in such manner as to yield a nearly flat initial term structure for the maturity range from 0 to 10 years (forward rates vary less than one base point). The Gaussian model was fitted to the same initial term structure and the Gaussian σ_j were chosen to match the factor variances with those in the CIR model[17]. Again, spread option prices are nearly equal. This result is remarkably stable, even when factor variances are matched for a different time horizon than the option

[15] see footnote 9

[16] For the compounding periods considered in our analysis below, comparisons between the closed–form solution for nominal spread options in the one–factor CIR case and the numerical results for continuous compounding show differences small enough to be ignored, as do comparisons between closed–form solutions for spread options on nominal and continuously compounded rates in the Gaussian models.

[17] For a complete parameter listing, see table D.

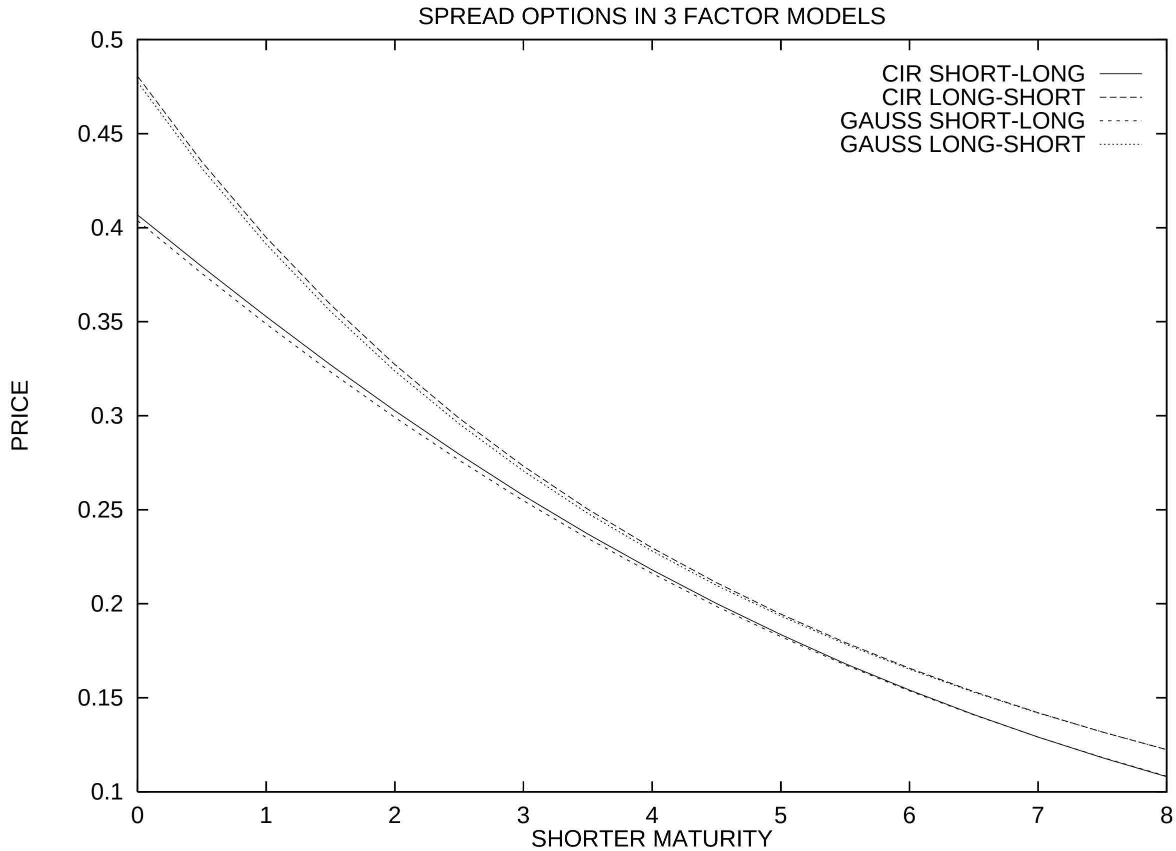

FIGURE 10. Spread options in three–factor models

maturity. Only for unrealistically high volatilities do marked differences appear between CIR and Gaussian spread option prices.

CONCLUSION

In the present paper, we have evaluated a broad collection of factor models of the term structure of interest rates proposed in the literature. They differ in the number of factors, in the class of probability distributions that the factors obey, and in whether or not mean reversion is included in the factor SDEs. Starting from a flat initial term structure of forward rates, we analysed the evolution of the shape of the term structure generated endogenously by the models.

In all the models considered, the introduction of mean reversion is essential in order to generate downward sloping forward rate curves in any substantial proportion. Furthermore, in Gaussian models choosing different rates of mean reversion for each factor is necessary to prevent collapse to the one–factor case. In contrast, the multifactor version of the Cox, Ingersoll jr. and Ross (1985) (CIR) model degenerates into the one–factor case if and only if mean reversion and volatility parameters are equal for all factors.

The choice of the number of sources of uncertainty driving a model is clearly more important for the evolution of the shape of the term structure than the distribution assumed for the factors. Independent of the distributional assumption, the models considered restrict the evolution of the shape of the term structure in much the same manner. This is true not only qualitatively, but also quantitatively, as evidenced by our comparison of spread option prices. Hence, if some observed term structure shape cannot be explained by one model, then it cannot be explained by the others either.

These findings have bearing on the use of term structure models for the pricing and hedging of certain derivatives. First, mean reversion is necessary in order to model a realistic term structure of volatilities. Second priority should be the choice of the "correct" number of factors, perhaps through principal components analysis[18]. This determines which shapes of the term structure can occur. The choice of distributional assumption is largely irrelevant when considering positions in fixed income instruments which depend primarily on the *relative* movements of interest rates for different maturities.

For some fixed income instruments, however, there are theoretical arguments against certain distributional assumptions that cannot be safely ignored. Rogers (1996) demonstrates how permitting negative interest rates, as Gaussian models do, leads to implausible valuation results for certain contingent claims such as long term zero coupon bonds and zero strike floors. Hogan and Weintraub (1993) show that models with lognormal continuously compounded short rates assign infinite values to Eurodollar futures, a result that also holds for β–root models with β greater than unity.

In view of the different methods for empirical testing outlined in the introduction, the results in this paper suggest the following conclusions. While taking into account the movements of the entire term structure is clearly important in order to determine empirically the number of factors necessary to adequately model interest rate dynamics, using this data to test for the best specification of the diffusion process may lead to rather unstable results in the sense that deviations of the realized term structure shapes from those permitted by the models will be the same across models and this will dominate any differences due to the distributional assumptions on the factors. Instead, to improve the selectivity of the test, one could concentrate on estimating the factor distributions with the observed factor dynamics. In one–factor short rate models this would be an argument for the approach taken by Chan, Karolyi, Longstaff and Sanders (1992), who focus on the dynamics of the continuously compounded short rate. For the affine multifactor case, one could make use of the fact that these models can be reparameterized in factors which are yields[19] (or forward yields, for that matter). The dynamics of these yield factors can then be estimated from observables, and the stability of these estimates under reparameterization to yield factors of different maturities is a criterion for evaluating the model.

Appendix A. Proof of proposition 2

Instantaneous forward rates are equal to minus the derivative with respect to maturity of the logarithmic zero coupon bond prices. Thus, from (7) with $n = 1$ we get

(15)
$$r_c(t,T) = \frac{2\theta}{\sigma^2}\left(wc(c+a)e^{c(T-t)} - \frac{1}{2}(c+a)\right) + 2wce^{c(T-t)}r(t)(1 - w(c+a)(e^{c(T-t)} - 1))$$

and once more taking the derivative with respect to maturity

$$\partial_2 r_c(t,T) = 4c^2w^2e^{c(T-t)}(\theta + c(1 - 2(c+a)we^{c(T-t)})r(t)) \tag{16}$$

Setting $\partial_2 r_c(t,T) = 0$ yields

$$e^{c(T-t)} = \frac{(c-a)(\theta + cr(t))}{(c+a)(-\theta + cr(t))} \tag{17}$$

[18]See for example Litterman and Scheinkman (1991).
[19]see Duffie and Kan (1992, 1996).

The derivative of (16) with respect to maturity is

$$
\begin{aligned}
\partial_2 r_c(t,T) \quad = \quad & \left(2\,c^3\,e^{c\,(T-t)}\left(-a+c+a\,e^{c\,(T-t)}+c\,e^{c\,(T-t)}\right)^{-4}\sigma^{-2}\right. \\
& \cdot\left(-a^4\,b+2\,a^3\,b\,c-2\,a\,b\,c^3+b\,c^4+a^4\,b\,e^{2\,c\,(T-t)}+2\,a^3\,b\,c\,e^{2\,c\,(T-t)}\right. \\
& \quad -2\,a\,b\,c^3\,e^{2\,c\,(T-t)}-b\,c^4\,e^{2\,c\,(T-t)}+2\,a^2\,c\,r(t)\,\sigma^2-4\,a\,c^2\,r(t)\,\sigma^2 \\
& \quad +2\,c^3\,r(t)\,\sigma^2+8\,a^2\,c\,e^{c\,(T-t)}\,r(t)\,\sigma^2-8\,c^3\,e^{c\,(T-t)}\,r(t)\,\sigma^2 \\
& \quad \left.+2\,a^2\,c\,e^{2\,c\,(T-t)}\,r(t)\,\sigma^2+4\,a\,c^2\,e^{2\,c\,(-t+T)}\,r(t)\,\sigma^2+2\,c^3\,e^{2\,c\,(T-t)}\,r(t)\,\sigma^2\right)
\end{aligned} \tag{18}
$$

Inserting (17) into (18), we get

$$\frac{-(b-c\,r(t))^2\,(b+c\,r(t))^2}{4\,r(t)^3\,\sigma^2}<0$$

and thus the forward rate curve has a local maximum at the maturity T_0 satisfying (17), and is upward sloping for shorter and downward sloping for longer maturities. Note that for $r(t)=\theta/a$, i.e. when the drift of the short rate process is zero, $T_0=t$, which means that the term structure is downward sloping for all maturities if the short rate realization is $r(t)\geq\theta/a$. For $r(t)\leq\frac{b}{\sqrt{a^2+2\sigma^2}}$, (17) has no solution, and T_0 goes to infinity as $r(t)\downarrow\frac{b}{\sqrt{a^2+2\sigma^2}}$. Therefore the term structure is upward sloping for all maturities if $r(t)\leq\frac{b}{\sqrt{a^2+2\sigma^2}}$.

Appendix B. Spread Options on Instantaneous Forward Rates in the Vasicek Model

As El Karoui, Lepage, Myneni, Roseau and Viswanathan (1991) show, we have

$$r(t)=r_c(0,t)+\int_0^t e^{-a(t-s)}\sigma dW_{Q^T}(s) \tag{19}$$

and therefore under the forward measure $r(t)$ is normally distributed with mean $\mu_{r(t)}=r_c(0,t)$ and variance $\sigma^2_{r(t)}=(1-e^{-2at})\sigma^2/2a$. Inserting (19) into (13) and (14), respectively, we see that under Q^T $r_c(t,T)$ is normally distributed with

$$
\begin{aligned}
\mu_{r_c(t,T)} &= r_c(0,T)+\frac{\sigma^2}{2a^2}(1-e^{-2at})(e^{-a(T-t)}-e^{-2a(T-t)}) \\
\sigma^2_{r_c(t,T)} &= e^{-2a(T-t)}(1-e^{-2at})\frac{\sigma^2}{2a}
\end{aligned}
$$

and $S_t(\tau)$ is normally distributed with

$$
\begin{aligned}
\mu_{S_t(\tau)} &= \frac{\partial r_c(0,T)}{\partial\tau}+\frac{\sigma^2}{2a}(1-e^{-2at})(2e^{-a(T-t)}-1)e^{-a(T-t)} \\
\sigma^2_{S_t(\tau)} &= e^{-2a(T-t)}(1-e^{-2at})\frac{\sigma^2 a}{2}
\end{aligned}
$$

Proposition 3. *Consider a European spread option whose payoff at time T is defined as*

$$[r_c(T,T+x)-r_c(T,T+y)]^+$$

The value $V_0([r_c(T,T+x)-r_c(T,T+y)]^+)$ at time 0 of this contingent claim is

$$V_0([r_c(T,T+x)-r_c(T,T+y)]^+)=B(0,T)\left(\mu_s\mathcal{N}\left(\frac{\mu_s}{\sigma_s}\right)+\sigma_s\varphi\left(\frac{\mu_s}{\sigma_s}\right)\right)$$

for $x > y$ (long − short), and for $x < y$ (short − long) we have

$$V_0([r_c(T, T+x) - r_c(T, T+y)]^+) = B(0,T)\left(\mu_s \mathcal{N}\left(\frac{\mu_s}{\sigma_s}\right) + \sigma_s \varphi\left(-\frac{\mu_s}{\sigma_s}\right)\right)$$

with

$$\mu_s := r_c(0, T+x) - r_c(0, T+y) + \frac{\sigma^2}{2a^2}(1 - e^{-2aT})(e^{-ax} - e^{-2ax} - (e^{-ay} - e^{-2ay}))$$
$$\sigma_s := \frac{\sigma^2}{a}(1 - e^{-2aT})\left(\frac{1}{2}(e^{-2ax} + e^{-2ay}) - e^{-a(x+y)}\right)$$

where $\mathcal{N}$ is the standard normal distribution function and φ the corresponding probability density.

Proof: Defining $J := r_c(T, T+x) - r_c(T, T+y)$ and following the approach taken in Jamshidian (1987), we can write

$$\begin{aligned}
&V_0([r_c(T, T+x) - r_c(T, T+y)]^+) \\
=&B(0,T)\int_0^\infty J \frac{1}{\sqrt{2\pi\sigma_s^2}} exp\left\{-\frac{1}{2}\left(\frac{J-\mu_s}{\sigma_s}\right)^2\right\} dJ \\
=&B(0,T)\frac{1}{\sqrt{2\pi\sigma_s^2}}\left[\left[-\sigma_s^2 \exp\left\{-\frac{1}{2}\left(\frac{J-\mu_s}{\sigma_s}\right)^2\right\}\right]_0^\infty + \mu_s \int_0^\infty \exp\left\{-\frac{1}{2}\left(\frac{J-\mu_s}{\sigma_s}\right)^2\right\} dJ\right] \\
=&B(0,T)\left[\mu_s \mathcal{N}\left(\frac{\mu_s}{\sigma_s}\right) + \sigma_s \varphi\left(\frac{\mu_s}{\sigma_s}\right)\right]
\end{aligned}$$

for $x > y$ (long − short), and for $x < y$ (short − long) we have

$$\begin{aligned}
&V_0([r_c(T, T+x) - r_c(T, T+y)]^+) \\
=&-B(0,T)\int_{-\infty}^0 J \frac{1}{\sqrt{2\pi\sigma_s^2}} \exp\left\{-\frac{1}{2}\left(\frac{J-\mu_s}{\sigma_s}\right)^2\right\} dJ \\
=&-B(0,T)\left[\mu_s \mathcal{N}\left(-\frac{\mu_s}{\sigma_s}\right) - \sigma_s \varphi\left(\frac{\mu_s}{\sigma_s}\right)\right]
\end{aligned}$$

□

Proposition 4. *For every fixed time to maturity difference $c > 0$ there exists a unique time to maturity $\overline{y}(a)$ such that*

$$V_0([r_c(T, T+y+c) - r_c(T, T+y)]^+) \gtrless V_0([r_c(T, T+y) - r_c(T, T+y+c)]^+)$$

as $y \lessgtr \overline{y}(a)$. Furthermore $\frac{d\overline{y}(a)}{da} < 0$.

Proof: We have

$$V_0([r_c(T, T+y+c) - r_c(T, T+y)]^+) - V_0([r_c(T, T+y) - r_c(T, T+y+c)]^+) = \mu_s$$

μ_s has the following properties:

i) $\lim_{y\to 0} \mu_s = \frac{\sigma^2}{2a^2}(1 - \exp\{-2aT_0\})\exp\{-ac\}(1 - \exp\{-ac\}) > 0$

ii) $\lim_{y\to\infty} \mu_s = 0$

iii) $\mu_s|_{\overline{y}} = 0 \qquad \Leftrightarrow \qquad \overline{y} = \frac{1}{a}\ln(\exp\{-ac\} + 1) > 0$

iv) $\left.\frac{d\mu_s}{dy}\right|_{\overline{\overline{y}}} = 0 \quad \Leftrightarrow \quad \overline{\overline{y}} = \frac{1}{a}\ln\left(2\exp\{-ac\}+1\right) > \frac{1}{a}\ln\left(\exp\{-ac\}+1\right)$

□

Proposition 4 shows that the Vasicek model favours upward sloping term structures on the short end and downward sloping term structures on the long end. For the Ho/Lee model, the slope of the forward rate curve and thus spread options prices are deterministic; starting from a flat initial term structure, only options on upward sloping forward rate curves will have a positive value.

Appendix C. Spread options on nominal forward rates

C.1. **In the one-factor Gaussian model.** Consider an European option with the payoff[20]

$$[r_n(T,x,\alpha) - r_n(T,y,\alpha)]^+ \tag{20}$$

at maturity T, where $r_n(T,x,\alpha)$ denotes the nominal forward rate at time T for the investment period from $T+x$ to $T+x+\alpha$. Following the approach taken in Jamshidian (1987), we write the price at time 0 of the option as

$$C(r,0) = \frac{1}{\alpha}B(r,t,T)(J(T,x,\alpha) - J(T,y,\alpha))$$

with

$$J(T,x,\alpha) = \int_{r(T)\in R_T^*} F(r,T,T+x,T+x+\alpha)^{-1}dQ^T(r(T))$$

where

$$R_T^* = \{r(T)\,|F(r,T,T+x,T+x+\alpha) < F(r,T,T+y,T+y+\alpha)\}$$

The forward price at time T of a forward contract maturing in $T+x$ on a zero coupon bond maturing in $T+x+\alpha$, parametrized in the spot short rate $r(T)$, is given by

$$F(r,T,T+x,T+x+\alpha) = F(0,T+x,T+x+\alpha)k_1k_2\exp\left\{-r(T)\frac{1}{a}(e^{-ax} - e^{-a(x+\alpha)})\right\}$$

with

$$k_1 = \exp\left\{\frac{-\sigma^2}{2a^2}(1-e^{-2aT})\left((\frac{1}{a}(e^{-ax} - e^{-a(x+\alpha)})) - \frac{1}{2a}(e^{-2ax} - e^{-2a(x+\alpha)})\right)\right\}$$

$$k_2 = \exp\left\{r_c(0,T)\frac{1}{a}(e^{-ax} - e^{-a(x+\alpha)})\right\}$$

As can be readily seen, the difference $F(r,T,T+x,T+x+\alpha) - F(r,T,T+y,T+y+\alpha)$ is monotonic in $r(T)$. Thus we have for $x > y$

$$J(T,x,\alpha) = F(0,T+x,T+x+\alpha)^{-1}k_1^{-1}k_2^{-1}\int_{-\infty}^{r_T^*}\exp\left\{r(T)\frac{1}{a}(e^{-ax} - e^{-a(x+\alpha)})\right\}dQ^T(r(T))$$

and for $x < y$

$$J(T,x,\alpha) = F(0,T+x,T+x+\alpha)^{-1}k_1^{-1}k_2^{-1}\int_{r_T^*}^{\infty}\exp\left\{r(T)\frac{1}{a}(e^{-ax} - e^{-a(x+\alpha)})\right\}dQ^T(r(T))$$

[20] For a treatment of this contingent claim within the context of an international economy see Frey and Sommer (1997).

r_T^* is given by the condition

$$F(r_T^*, T, T+x, T+x+\alpha) = F(r_T^*, T, T+y, T+y+\alpha)$$

$$\iff r_T^* = \left(a \ \ln \frac{F(0,T+x,T+x+\alpha)}{F(0,T+y,T+y+\alpha)} - \frac{1}{2a^2}\sigma^2_{r(T)}(k_3 - k_4)\right) k_3^{-1} + r_c(0,T)$$

with

$$k_3 = e^{-ax} - e^{-ay} - e^{-a(x+\alpha)} + e^{-a(y+\alpha)}$$

$$k_4 = \frac{1}{2}\left(e^{-2ax} - e^{-2ay} - e^{-2a(x+\alpha)} + e^{-2a(y+\alpha)}\right)$$

and $\sigma^2_{r(T)}$, the variance of the spot short rate at time T, viewed from time 0, given by

$$\sigma^2_{r(T)} = \frac{\sigma^2}{2a}\left(1 - e^{-2aT}\right)$$

$r(T)$ is normally distributed with variance $\sigma^2_{r(T)}$ and mean $\mu^Q_{r(T)} = r_c(0,T)$ under the forward risk-adjusted measure Q^T. Therefore

$$\int_{-\infty}^{r_T^*} \exp\left\{r(T)\frac{1}{a}(e^{-ax} - e^{-a(x+\alpha)})\right\} dQ^T(r(T))$$

$$= \int_{-\infty}^{r_T^*} \exp\left\{r(T)\frac{1}{a}(e^{-ax} - e^{-a(x+\alpha)})\right\} \frac{1}{\sqrt{2\pi}\sigma_{r(T)}} \ \exp\left\{-\frac{(r(T)-\mu^Q_{r(T)})^2}{2\sigma^2_{r(T)}}\right\} dr(T)$$

$$= \exp\left\{\frac{-(\mu^Q_{r(T)})^2 + (\mu^Q_{r(T)} + \frac{1}{a}(e^{-ax} - e^{-a(x+\alpha)})\sigma^2_{r(T)})^2}{2\sigma^2_{r(T)}}\right\}$$

$$\cdot \int_{-\infty}^{r_T^*} \frac{1}{\sqrt{2\pi}\sigma_r(T)} \ \exp\left\{-\frac{(r(T) - ((\mu^Q_{r(T)}) + \frac{1}{a}(e^{-ax} - e^{-a(x+\alpha)})\sigma^2_{r(T)}))^2}{2\sigma^2_{r(T)}}\right\} dr(T)$$

$$= \exp\left\{\frac{1}{a}(e^{-ax} - e^{-a(x+\alpha)})r_c(0,T) + \frac{1}{2a^2}(e^{-ax} - e^{-a(x+\alpha)})^2\sigma^2_{r(T)}\right\}$$

$$\cdot N\left(\frac{r^*(T) - r_c(0,T) - \frac{1}{a}(e^{-ax} - e^{-a(x+\alpha)})\sigma^2_{r(T)}}{\sigma_{r(T)}}\right)$$

C.2. **Spread options in the one-factor "square root" term structure model.** Writing (20) in terms of zero coupon bond prices and substituting these with the bondprice formula given by Jamshidian (1987), we get

$$\frac{1}{\alpha}\left[\frac{B(r,T,T+x)}{B(r,T,T+x+\alpha)} - \frac{B(r,T,T+y)}{B(r,T,T+y+\alpha)}\right]^+$$

$$= \frac{1}{\alpha}\left[\frac{\mathcal{A}(T,T+x)}{\mathcal{A}(T,T+x+\alpha)}\exp\left\{-r(T)\left(\mathcal{B}(T,T+x) - \mathcal{B}(T,T+x+\alpha)\right)\right\}\right.$$

$$\left. - \frac{\mathcal{A}(T,T+y)}{\mathcal{A}(T,T+y+\alpha)}\exp\left\{-r(T)(\mathcal{B}(T,T+y) - \mathcal{B}(T,T+y+\alpha))\right\}\right]^+$$

Again, it is easy to show that the difference $F(r,T,T+x,T+x+\alpha) - F(r,T,T+y,T+y+\alpha)$ is monotone in $r(T)$. Thus, following Jamshidian's approach, the price in t of the option is given by

$$C(r,t) = \frac{1}{\alpha}B(r,t,T)\left(J(T,x,\alpha) - J(T,y,\alpha)\right)$$

with

$$J(T,x,\alpha) := \int_0^{r_T^*} \frac{\mathcal{A}(T,T+x)}{\mathcal{A}(T,T+x+\alpha)} \exp\{-r(T)(\mathcal{B}(T,T+x) - \mathcal{B}(T,T+x+\alpha)\} dQ^T(r(T))$$

for $x > y$, and

$$J(T,x,\alpha) := \int_{r_T^*}^{\infty} \frac{\mathcal{A}(T,T+x)}{\mathcal{A}(T,T+x+\alpha)} \exp\{-r(T)(\mathcal{B}(T,T+x) - \mathcal{B}(T,T+x+\alpha)\} dQ^T(r(T))$$

for $x < y$, where

$$r_T^* = \frac{\ln \mathcal{A}(T,T+x) - \ln \mathcal{A}(T,T+x+\alpha) - \ln \mathcal{A}(T,T+y) + \ln \mathcal{A}(T,T+y+\alpha)}{\mathcal{B}(T,T+x) - \mathcal{B}(T,T+x+\alpha) - \mathcal{B}(T,T+y) + \mathcal{B}(T,T+y+\alpha)}$$

Let

$$\tilde{r}(T) := r(T) \cdot 2(w\sigma^2(e^{c(T-t)} - 1))^{-1} = \frac{4}{\sigma^2}\mathcal{B}(t,T)^{-1} r(T)$$

Then we know that $\tilde{r}(T)$ conditioned on $r(t)$ is noncentral chi-square distributed under the forward risk adjusted measure Q^T, with ν degrees of freedom and non-centrality parameter λ given by

$$\begin{aligned} \nu &= \frac{4\theta}{\sigma^2} \\ \lambda &= \frac{16 w^2 c^2 e^{c(T-t)}}{\sigma^2 \mathcal{B}(t,T)} r(t) \end{aligned} \tag{21}$$

Let

$$\begin{aligned} b &:= \frac{4}{\sigma^2}\mathcal{B}(t,T)^{-1} \\ L &:= \mathcal{B}(T,T+x) - \mathcal{B}(T,T+x+\alpha) \end{aligned} \tag{22}$$

Then we have

$$J(T,x,\alpha) = b \int_0^{r_T^*} \frac{\mathcal{A}(T,T+x)}{\mathcal{A}(T,T+x+\alpha)} e^{-r(T)L} q_{\chi^2}(b\ r(T),\nu,\lambda) dr(T)$$

Substituting for $q_{\chi^2}^T$ its infinite sum expression (see Johnson and Kotz (1970), Chapter 28, eq. 3), we get

$$\begin{aligned}
& e^{-r(T)L} q_{\chi^2}(b\ r(T),\nu,\lambda) \\
= \quad & e^{-r(T)L} 2^{-\frac{1}{2}\nu} \exp\left\{-\frac{1}{2}(b\ r(t) + \lambda)\right\} \sum_{j=0}^{\infty} \frac{(b\ r(t))^{\frac{1}{2}\nu+j-1}\lambda^j}{\Gamma(\frac{1}{2}\nu+j)2^{2j}j!} \\
= \quad & 2^{-\frac{1}{2}\nu} \exp\left\{-\frac{1}{2}\left(r(T)(b+2L) + \frac{\lambda b}{b+2L}\right)\right\} \exp\left\{\frac{1}{2}\left(\frac{\lambda b}{b+2L} - \lambda\right)\right\} \left(\frac{b}{b+2L}\right)^{\frac{1}{2}\nu-1} \\
& \cdot \sum_{j=0}^{\infty} \frac{((b+2L)r(T))^{\frac{1}{2}\nu+j-1}\left(\frac{\lambda b}{b+2L}\right)^j}{\Gamma(\frac{1}{2}\nu+j)2^{2j}j!} \\
= \quad & \exp\left\{\frac{1}{2}\left(\frac{\lambda b}{b+2L} - \lambda\right)\right\} \left(\frac{b}{b+2L}\right)^{\frac{1}{2}\nu-1} q_{\chi^2}\left((b+2L)r(T),\nu,\frac{\lambda b}{b+2L}\right)
\end{aligned}$$

Therefore

$$J(T,x,\alpha) = b \cdot \exp\left\{\frac{1}{2}\left(\frac{\lambda b}{b+2L} - \lambda\right)\right\}\left(\frac{b}{b+2L}\right)^{\frac{1}{2}\nu - 1} \frac{\mathcal{A}(T,T+x)}{\mathcal{A}(T,T+x+\alpha)}$$
$$\cdot \int_0^{r_T^*} q_{\chi^2}\left((b+2L)r(T), \nu, \frac{\lambda b}{b+2L}\right) dr(T)$$
$$= \exp\left\{\frac{1}{2}\left(\frac{\lambda b}{b+2L} - \lambda\right)\right\}\left(\frac{b}{b+2L}\right)^{\frac{1}{2}\nu} \frac{\mathcal{A}(T,T+x)}{\mathcal{A}(T,T+x+\alpha)}$$
$$\cdot \int_0^{r_T^*} q_{\chi^2}\left((b+2L)r(T), \nu, \frac{\lambda b}{b+2L}\right) d\left((b+2L)r(T)\right)$$

for $x > y$, and for $x < y$

$$J(T,x,\alpha) = \exp\left\{\frac{1}{2}\left(\frac{\lambda b}{b+2L} - \lambda\right)\right\}\left(\frac{b}{b+2L}\right)^{\frac{1}{2}\nu} \frac{\mathcal{A}(T,T+x)}{\mathcal{A}(T,T+x+\alpha)}$$
$$\cdot \left(1 - \int_0^{r_T^*} q_{\chi^2}\left((b+2L)r(T), \nu, \frac{\lambda b}{b+2L}\right) d\left((b+2L)r(T)\right)\right)$$

with

$$\int_0^{r_T^*} q_{\chi^2}\left((b+2L)r(T), \nu, \frac{\lambda b}{b+2L}\right) d\left((b+2L)r(T)\right) = \chi^2_{\nu, \frac{\lambda b}{b+2L}}\left((b+2L)r_T^*\right)$$

the value of the non-central chi-square distribution function.

C.3. **Spread options in the multi-factor square root term structure model.** Writing (20) in terms of zero coupon bond prices and substituting these with the bond price formula given in Chen and Scott (1995), we get

$$\frac{1}{\alpha}\left[\frac{B(z,T,T+x)}{B(z,T,T+x+\alpha)} - \frac{B(z,T,T+y)}{B(z,T,T+y+\alpha)}\right]^+$$
$$= \frac{1}{\alpha}\left[\prod_{j=1}^{n} \frac{\mathcal{A}_j(T,T+x)}{\mathcal{A}_j(T,T+x+\alpha)} \exp\{-(\mathcal{B}_j(T,T+x) - \mathcal{B}_j(T,T+x+\alpha))z_j(T)\} - \right.$$
$$\left.\prod_{j=1}^{n} \frac{\mathcal{A}_j(T,T+y)}{\mathcal{A}_j(T,T+y+\alpha)} \exp\{-(\mathcal{B}_j(T,T+y) - \mathcal{B}_j(T,T+y+\alpha))z_j(T)\}\right]^+ .$$

Again following the approach taken in Jamshidian (1987), the price in t of the option is given by

$$C(z,t) = \frac{1}{\alpha} B(z,t,T)(J(T,x,\alpha) - J(T,y,\alpha))$$

with J given by the multidimensional integral

$$J(T,x,\alpha)$$
$$:= \int_{z(T)\in Z_T^*} \left(\prod_{j=1}^{n} \frac{\mathcal{A}_j(T,T+x)}{\mathcal{A}_j(T,T+x+\alpha)} \exp\{-(\mathcal{B}_j(T,T+x) - \mathcal{B}_j(T,T+x+\alpha))z_j(T)\}\right) dQ^T(z(T))$$

and

$$Z_T^* = \left\{ z(T) \left| \sum_{j=1}^{n} (\mathcal{B}_j(T,T+y) - \mathcal{B}_j(T,T+y+\alpha) - \mathcal{B}_j(T,T+x) + \mathcal{B}_j(T,T+x+\alpha)) z_j(T) \right. \right.$$
$$\left. > \ln \left(\prod_{j=1}^{n} \frac{\mathcal{A}_j(T,T+y)}{\mathcal{A}_j(T,T+y+\alpha)} \frac{\mathcal{A}_j(T,T+x+\alpha)}{\mathcal{A}_j(T,T+x)} \right) \right\}$$

Now consider

$$\exp\{-(\mathcal{B}_j(T,T+x) - \mathcal{B}_j(T,T+x+\alpha)) z_j(T)\} dQ^T(z_j(T)) \tag{23}$$

Analogously to the one-factor case, defining ν_j and λ_j as in (21), b_j and L_j as in (22), we can write (23) as

$$\begin{aligned} &\exp\{-L_j z_j(T)\}\, q_{\chi^2}(b_j z_j(T), \nu_j, \lambda_j) dz_j(T) \\ =&\exp\left\{ \frac{1}{2}\left(\frac{\lambda_j b_j}{b_j + 2L_j} - \lambda_j \right) \right\} \left(\frac{b_j}{b_j+2L_j} \right)^{\frac{1}{2}\nu_j - 1} q_{\chi^2}\left((b_j + 2L_j) z_j(T), \nu_j, \frac{\lambda_j b_j}{b_j + 2L_j} \right) dz_j(T) \end{aligned}$$

Because of the independence of factors, we can write

$$\begin{aligned} J(T,x,\alpha) &= \left(\prod_{j=1}^{n} \frac{\mathcal{A}_j(T,T+x)}{\mathcal{A}_j(T,T+x+\alpha)} \right) \int_{z(T)\in Z_T^*} \left(\prod_{j=1}^{n} b_j e^{-L_j z_j(T)} q_{\chi^2}(b_j z_j(T), \nu_j, \lambda_j) dz_j(T) \right) \\ &= \left(\prod_{j=1}^{n} \frac{\mathcal{A}_j(T,T+x)}{\mathcal{A}_j(T,T+x+\alpha)} \exp\left\{ \frac{1}{2}\left(\frac{\lambda_j b_j}{b_j+2L_j} - \lambda_j \right) \right\} \left(\frac{b_j}{b_j+2L_j} \right)^{\frac{1}{2}\nu_j} \right) \\ &\quad \cdot \int_{z(T)\in Z_T^*} \left(\prod_{j=1}^{n} q_{\chi^2}\left((b_j+2L_j) z_j(T), \nu_j, \frac{\lambda_j b_j}{b_j+2L_j} \right) d((b_j - 2L_j) z_j(T)) \right) \end{aligned}$$

The remaining integral term is the value of the probability function of a linear combination of independent random variables which are non-central chi-square distributed, and can be evaluated using the techniques described by Chen and Scott (1995):

$$Z_T^* = \left\{ z(T) \left| \sum_{j=1}^{n} \beta_{xj} \hat{z}_{xj}(T) > k \right. \right\}$$

with

$$\begin{aligned} \hat{z}_{xj}(T) &:= (b_j + 2L_j) z_j(T) \\ \beta_j &:= (\mathcal{B}_j(T,T+y) - \mathcal{B}_j(T,T+y+\alpha) - \mathcal{B}_j(T,T+x) + \mathcal{B}_j(T,T+x+\alpha))(b_j + 2L_j)^{-1} \\ k &:= \ln \left(\prod_{j=1}^{n} \frac{\mathcal{A}_j(T,T+y)}{\mathcal{A}_j(T,T+y+\alpha)} \frac{\mathcal{A}_j(T,T+x+\alpha)}{\mathcal{A}_j(T,T+x)} \right) \end{aligned}$$

and so we can write

$$\int_{z(T)\in Z_T^*} \left(\prod_{j=1}^{n} q_{\chi^2}\left((b_j+2L_j) z_j(T), \nu_j, \frac{\lambda_j b_j}{b_j+2L_j} \right) d((b_j-2L_j) z_j(T)) \right) = 1 - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin uk}{u} \Psi(u) du$$

where

$$\Psi(u) = \prod_{j=1}^{n} \Psi_j(\beta_{xj} u)$$

with

$$\Psi_j(\beta_{xj}u) = (1-2i\beta_{xj}u)^{-\frac{1}{2}\nu_j}\exp\left\{\frac{i\beta_{xj}u}{1-2i\beta_{xj}u}\frac{\lambda_j b_j}{b_j+2L_j}\right\}.$$

We then evaluate the integral numerically, as in Chen and Scott (1995).

C.4. **Spread options in the multi–factor Gaussian term structure model.** As before, we write the price in t of the option as

$$C(z,t) = \frac{1}{\alpha}B(z,t,T)\left(J(T,x,\alpha) - J(T,y,\alpha)\right).$$

Defining

$$\xi_j(t,T,\alpha) := \int_t^T \frac{\sigma_j}{a_j}\left(e^{-a_j(T-s)} - e^{-a_j(T+\alpha-s)}\right)dW_j^{Q^T}(s) \tag{24}$$

we have $\xi_j(t,T,\alpha)$ normally distributed with

$$\begin{aligned}
\mathrm{E}_t^{Q^T}\left[\xi_j(t,T,\alpha)\right] &= 0\\
\sigma^2_{\xi_j} := \mathrm{Var}_t^{Q^T}\left[\xi_j(t,T,\alpha)\right] &= \int_t^T \frac{\sigma_j^2}{a_j^2}\left(e^{-a_j(T-s)} - e^{-a_j(T+\alpha-s)}\right)^2 ds.
\end{aligned}$$

As in the previous section,

$$J(T,x,\alpha) := \int_{z(T)\in Z_T^*} \frac{B(z,T,T+x)}{B(z,T,T+x+\alpha)}dQ^T(z(T)) \tag{25}$$

with Z_T^* suitably defined. Substituting (11) into (25) and writing the equation in terms of ξ_j, we get

$$\begin{aligned}
J(T,x,\alpha) =& \frac{B(z,T,T+x)}{B(z,T,T+x+\alpha)}\\
&\cdot\exp\left\{-\frac{1}{2}\sum_{j=1}^n\int_t^T\frac{\sigma_j^2}{a_j^2}\left(\left(1-e^{-a_j(T+x-s)}\right)^2 - \left(1-e^{-a_j(T+x+\alpha-s)}\right)^2\right)ds\right\}\\
&\cdot\exp\left\{-\sum_{j=1}^n\int_t^T\frac{\sigma_j^2}{a_j^2}\left(1-e^{-a_j(T-s)}\right)\left(e^{-a_j(T+x-s)} - e^{-a_j(T+x+\alpha-s)}\right)ds\right\}\\
&\cdot\int_{\xi(t,T,\alpha)\in\Xi_T^*}\exp\left\{\sum_{j=1}^n e^{-a_jx}\xi_j(t,T,\alpha)\right\}dQ^T\left(\xi_j(t,T,\alpha)\right)
\end{aligned} \tag{26}$$

where

$$\Xi_T^* := \Bigg\{ \xi(t,T,\alpha) \Bigg| \sum_{j=1}^{n} \left(e^{-a_j x} - e^{-a_j y}\right) \xi_j(t,T,\alpha) > \ln \frac{B(z,t,T+y)B(z,t,T+x+\alpha)}{B(z,t,T+y+\alpha)B(z,t,T+x)}$$

$$(27) \qquad - \sum_{j=1}^{n} \frac{\sigma_j^2}{2a_j^3} \Bigg(e^{-a_j y} \left(\frac{1}{2}\left(e^{-a_j y} - e^{-a_j(2(T-t)+y)}\right) - 1 + e^{-2a_j(T-t)} \right)$$

$$- e^{-a_j(y+\alpha)} \left(\frac{1}{2}\left(e^{-a_j(y+\alpha)} - e^{-a_j(2(T-t)+y+\alpha)}\right) - 1 + e^{-2a_j(T-t)} \right)$$

$$- e^{-a_j x} \left(\frac{1}{2}\left(e^{-a_j x} - e^{-a_j(2(T-t)+x)}\right) - 1 + e^{-2a_j(T-t)} \right)$$

$$+ e^{-a_j(x+\alpha)} \left(\frac{1}{2}\left(e^{-a_j(x+\alpha)} - e^{-a_j(2(T-t)+x+\alpha)}\right) - 1 + e^{-2a_j(T-t)} \right) \Bigg) \Bigg\}$$

We now proceed to calculate the truncated expectation in (26). Because the Brownian motions W_j^Q are independent, we can write

$$\int_{\xi(t,T,\alpha)\in\Xi_T^*} \exp\left\{ \sum_{j=1}^{n} e^{-a_j x} \xi_j(t,T,\alpha) \right\} dQ^T\left(\xi_j(t,T,\alpha)\right)$$

$$= \int_{\xi(t,T,\alpha)\in\Xi_T^*} \left(\prod_{j=1}^{n} \exp\left\{ e^{-a_j x}\xi_j(t,T,\alpha) \right\} \frac{1}{\sqrt{2\pi}\sigma_{\xi_j}} \exp\left\{ -\frac{\xi_j(t,T,\alpha)^2}{2\sigma_{\xi_j}^2} \right\} d\xi_j(t,T,\alpha) \right)$$

$$= \exp\left\{ \frac{1}{2}\sum_{j=1}^{n} e^{-2a_j x}\sigma_{\xi_j}^2 \right\} \int_{\xi(t,T,\alpha)\in\Xi_T^*} \left(\prod_{j=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_{\xi_j}} \exp\left\{ -\frac{1}{2\sigma_{\xi_j}^2} \left(\xi_j(t,T,\alpha) - e^{-a_j x}\sigma_{\xi_j}^2 \right)^2 \right\} d\xi_j(t,T,\alpha) \right)$$

$$= \exp\left\{ \frac{1}{2}\sum_{j=1}^{n} e^{-2a_j x}\sigma_{\xi_j}^2 \right\} \int_{\xi(t,T,\alpha)\in\Xi_T^*} \left(\prod_{j=1}^{n} q_N^T\left(\xi_j(t,T,\alpha)\right) d\xi_j(t,T,\alpha) \right)$$

where the $q_N^T(\cdot)$ are the appropriately defined densities of independent normal distributions. Therefore

$$\int_{\xi(t,T,\alpha)\in\Xi_T^*} \left(\prod_{j=1}^{n} q_N^T\left(\xi_j(t,T,\alpha)\right) d\xi_j(t,T,\alpha) \right) = \text{Prob}\left[\sum_{j=1}^{n} \beta_j \xi_j(t,T,\alpha) > k \right]$$

with

$$\beta_j := e^{-a_j x} - e^{-a_j y}$$

and k defined as the right hand side of the inequality in (27). Since the sum of normally distributed random variables is again normally distributed, we have

$$\text{Prob}\left[\sum_{j=1}^{n} \beta_j \xi_j(t,T,\alpha) > k \right] = 1 - \mathcal{N}\left(\frac{k - \sum_{j=1}^{n} \beta_j e^{-a_j x}\sigma_{\xi_j}^2}{\sqrt{\sum_{j=1}^{n} \beta_j^2 \sigma_{\xi_j}^2}} \right)$$

Appendix D. Parameter Constellations

D.1. **One–Factor Models.** The initial curve of instantaneous forward rates is flat at 6% for all plots.

Figure	Model	Time[21]	β	MR	σ	Range[22]	Ref.[23]	Dev.[24]
2	Brennan/Schwartz	2 yrs.	1	0.2	0.15	$(0;\infty)$	16	0.00158
4	Vasicek	2 yrs.	0	0.15	0.01	$(-\infty;\infty)$	16	0.00195
5	CIR	2 yrs.	0.5	0.15	0.0408914	$[0;\infty)$	16	0.00193
6	Brennan/Schwartz	2 yrs.	1	0.15	0.1649986	$(0;\infty)$	16	0.00192
7	β–root	2 yrs.	1.5	0.15	0.6578632	$(0;\infty)$	16	0.00146
8	BDT	2 yrs.	na	0.15	0.1436448	$(0;\infty)$	16	0.00145
9	SaSo	2 yrs.	na	na	0.1483548	$(0;\infty)$	16	0.00147

D.2. **Multifactor Models.**

Fig.	Model	Time[21]	a_1	a_2	a_3	σ_1	σ_2	σ_3	θ_1	θ_2	θ_3
1	CIR	2 yrs.	0.08	0.16	na	0.03	0.09	na	0.0025	0.0025	na
3	CIR	2 yrs.	0.15	0.15	na	0.03	0.09	na	0.00325	0.00325	na
10	CIR	2 yrs.	0.1	0.15	0.2	0.03	0.04	0.05	0.002607	0.0029996	0.003426
	Gauss	2 yrs.	0.1	0.15	0.2	0.002916	0.007256	0.006135	na	na	na

The initial state of the world for figure 10 is $z_1(0) = z_2(0) = z_3(0) = 0.02$, i.e. the initial short rate is 6%. The θ_j are chosen in such a manner that all initial forward rates up to a time horizon of 10 years lie with half a base point of 6%.

References

Amin, K. I. and Morton, A. J. (1994), Implied Volatility Functions in Arbitrage–free Term Structure Models, *Journal of Financial Economics* **35**, 141–180.

Arnold, L. (1973), *Stochastische Diffentialgleichungen*, R. Oldenbourg Verlag.

Black, F. and Karasinski, P. (1991), Bond and Option Pricing when Short Rates are Lognormal, *Financial Analysts Journal* pp. 52–59.

Brennan, M. J. and Schwartz, E. S. (1977), Savings Bonds, Retractable Bonds and Callable Bonds, *Journal of Financial Economics* pp. 67–88.

Brown, R. H. and Schaefer, S. M. (1994), The Term Structure of Real Interest Rates and the Cox, Ingersoll, and Ross Model, *Journal of Financial Economics* **35**(1), 3–42.

Chan, K. C., Karolyi, G. A., Longstaff, F. A. and Sanders, A. B. (1992), An Empirical Comparison of Alternative Models of the Short–Term Interest Rate, *The Journal of Finance* **XLVII**(3), 1209–1227.

Chen, R.-R. and Scott, L. (1993), Maximum Likelihood Estimation for a Multifactor Equilibrium Model of the Term Structure of Interest Rates, *The Journal of Fixed Income* **3**(3), 14–31.

Chen, R.-R. and Scott, L. (1995), Interest Rate Options in Multifactor Cox-Ingersoll-Ross Models of the Term Structure, *The Journal of Derivatives* **3**(2), 52–72.

Cohen, H. and Heath, D. (1992), A New Method of Testing Pricing Models As Applied to Forward Interest Rate Models, Federal Reserve Bank, Atlanta, working paper Working Paper 92–16.

Cox, J. C., Ingersoll jr., J. E. and Ross, S. A. (1985), A Theory of the Term Structure of Interest Rates, *Econometrica* **53**(2), 385–407.

[21] The point in time for which the possible term structure realizations are plotted. For options, the maturity of the option.

[22] Range of possible short rate realizations.

[23] Refinement: the discretization of the time line in periods per year.

[24] The maximum deviations of the calculated period 0 term structure from the input term structure (in base points). See section 2.2.

Duffie, J. D. and Kan, R. (1992), A Yield–Factor Model of Interest Rates, Stanford University, working paper.

Duffie, J. D. and Kan, R. (1996), A Yield Factor Model of Interest Rates, *Mathematical Finance* **6**(4), 379–406.

El Karoui, N. and Rochet, J.-C. (1989), A Pricing Formula for Options on Coupon Bonds, working paper.

El Karoui, N., Lepage, C., Myneni, R., Roseau, N. and Viswanathan, R. (1991), The Valuation and Hedging of Contingent Claims with Markovian Interest Rates, Université de Paris 6, working paper.

Flesaker, B. (1993), Testing the Heath–Jarrow–Morton/Ho–Lee Model of Interest Rate Contingent Claims Pricing, *Journal of Financial and Quantitative Analysis* **28**(4), 483–495.

Frey, R. and Sommer, D. (1997), A Systematic Approach to Pricing and Hedging International Derivatives with Interest Rate Risk, *Applied Mathematical Finance,* forthcoming.

Geman, H., El Karoui, N. and Rochet, J.-C. (1995), Changes of Numeraire, Changes of Probability Measure and Option Pricing, *Journal of Applied Probability* **32**, 443–458.

Heath, D., Jarrow, R. and Morton, A. (1992), Bond Pricing and the Term Structure of the Interest Rates: A New Methodology, *Econometrica* **60**(1), 77–105.

Ho, T. S. and Lee, S.-B. (1986), Term Structure Movements and Pricing Interest Rate Contingent Claims, *Journal of Finance* **XLI**(5), 1011–1029.

Hogan, M. and Weintraub, K. (1993), The Lognormal Interest Rate Model and Eurodollar Futures, Citibank, New York, working paper.

Hull, J. and White, A. (1990), Pricing Interest–Rate Derivative Securities, *The Review of Financial Studies* **3**(4), 573–592.

Hull, J. and White, A. (1993), One–Factor Interest–Rate Models and the Valuation of Interest–Rate Derivative Securities, *Journal of Financial and Quantitative Analysis* **28**(2), 235–254.

Jamshidian, F. (1987), Pricing of Contingent Claims in the One–Factor Term Structure Model, Financial Strategies Group, Merrill Lynch Capital Markets, working paper.

Johnson, N. L. and Kotz, S. (1970), *Continuous Univariate Distributions - 2,* The Houghton Mifflin Series in Statistics, John Wiley & Sons, Inc., New York, New York, USA.

Litterman, R. and Scheinkman, J. (1991), Common Factors Affecting Bond Returns, *The Journal of Fixed Income* **1**, 54–61.

Rogers, C. (1996), Gaussian Errors, *Risk Magazine* **9**(1), 42–45.

Rogers, L. C. G. (1995), Which model for Term–Structure of Interest Rates should one use?, *in* M. H. A. Davis, J. D. Duffie, W. H. Fleming and S. E. Shreve (eds), *Mathematical Finance*, Vol. 65 of *The IMA Volumes in Mathematics and Its Applications*, Springer, pp. 93–115.

Sandmann, K. and Sondermann, D. (1993), On the Stability of Lognormal Interest Rate Models, SFB 303, Universität Bonn, working paper B–263.

Schlögl, E. and Sommer, D. (1994), On Short Rate Processes and their Implications for Term Structure Movements, SFB 303, Universität Bonn, working paper B–293.

Stambaugh, R. F. (1988), The Information in Forward Rates: Implications for Models of the Term Structure, *Journal of Financial Economics* **21**(1), 41–70.

Vasicek, O. (1977), An Equilibrium Characterization of the Term Structure, *Journal of Financial Economics* **5**, 177–188.