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Quasimonotone Individual Demand

by

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Abstract

Quasimonotone individual demand correspondences are characterized as those which can be rationalized (in a weak sense) by a complete, upper continuous, monotone, and convex preference relation. Moreover, it is shown that an arbitrary set of demand observations can be rationalized by a reflexive, upper continuous, monotone and convex preference if and only if it is probperly quasimonotone.

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1 Revealed preferences and quasimonotonicity

In traditional microeconomics a consumer is typically described by a binary relation R (his *preference*) on the set $\mathbb{R}^l_+$ of all possible consumption bundles with l commodities. For $x, y \in \mathbb{R}^l_+$, xRy is interpreted as "x is at least as good as y" or "x is weakly preferred to y".

Usually, additional properties of R are required. Some of the most important ones are

Reflexivity:	For all $x \in \mathbb{R}^l_+ : xRx$.
Completeness:	For all $x, y \in \mathbb{R}^l_+ : xRy \vee yRx$.
Transitivity:	For all $x, y, z \in \mathbb{R}^l_+ : xRy \wedge yRz \Rightarrow xRz$.
Convexity:	For all $x \in \mathbb{R}^l_+ : R(x) = \{y \in \mathbb{R}^l_+ \mid yRx\}$ is convex.
Monotonicity:	For all $x, y \in \mathbb{R}^l_+ : y \gg x \Rightarrow \neg xRy$.
Local nonsatiation:	In every neighborhood U of $x \in \mathbb{R}^l_+$ there exists y such that $\neg xRy$.
Continuity:	R is closed in $\mathbb{R}^l_+ \times \mathbb{R}^l_+$.

Assume that the l commodity prices are given by a price vector $p \in \mathbb{R}^l_{++}$ and that the consumer's wealth is $w > 0$. The hypothesis is that he chooses a consumption bundle in his *budget set* $B(p, w) = \{x \in \mathbb{R}^l_+ | px \leq w\}$ that is weakly preferred to all elements in $B(p, w)$.

Since $B(p, w) = B(\lambda p, \lambda w)$ for all $\lambda > 0$, there is no loss of generality to describe the consumer's choice problem by restricting his wealth to be equal to one, i.e. his possible choice sets are given by

$$B(p) = \{x \in \mathbb{R}^l_+ | px \leq 1\}, p \in \mathbb{R}^l_{++}.$$

Thus, any preference R induces a demand relation $D_R \subseteq \mathbb{R}^l_{++} \times \mathbb{R}^l_+$ defined by

$$(p, x) \in D_R \text{ iff } x \in B(p) \text{ and } xRy \text{ for all } y \in B(p).$$

Since only subsets D of D_R are observable (and not the preference R itself), revealed preference theory tries to relate assumptions on R with conditions on $D \subseteq D_R$. Two typical questions are the following. What are the restrictions on D_R implied by certain properties of R? Which properties of R are revealed by an (appropriately defined) "consistent" demand behavior?

It is easy to see that even if completeness, transitivity, convexity, and continuity are all required to be satisfied by R, there is no restriction on demand. Indeed, $D_R = \{(p,x)|px \leq 1\}$ for $R = \mathbb{R}^l_+ \times \mathbb{R}^l_+$!

If, to avoid such trivialities, R is locally nonsatiated (or even monotone), then, obviously, $(p,x) \in D_R$ implies the budget identity $px = 1$. Conversely, any D with $px = 1$ for all $(p,x) \in D$ can be obtained as a subset of D_R for some R which is monotone, complete, and continuous (define xRy by $\neg y \gg x$).

In the sequel, only locally nonsatiated preferences are considered.

Accordingly, we call $D \subseteq \mathbb{R}^l_{++} \times \mathbb{R}^l_+$ a *demand relation* if $px = 1$ for all $(p,x) \in D$.

We say that D is (weakly) *rationalized* by a preference R if $D \subseteq D_R$.

In his seminal contribution, Samuelson (1938)[1] introduced the following consistency postulate for demand functions which is now well known as the "Weak Axiom of Revealed Preference":

$$\text{For all } (p,x),(q,y) \in D \text{ such that } x \neq y:$$
$$p(y-x) \leq 0 \text{ implies } q(y-x) < 0.$$

It can be justified by a simple argument. Observe that $p(y-x) \leq 0$ and $(p,x) \in D$ means that x has been chosen at p while y could have been chosen. If, in case that $x \neq y$, this is interpreted as "x is revealed preferred to y" then consistent behavior should imply that y must not be revealed preferred to x, i.e. $q(x-y) > 0$ for $(q,y) \in D$.

However, this interpretation is very strong since it excludes that the consumer is indifferent between two optimal choices. As an alternative, we call x *to be revealed preferred* to y if there is $p \in \mathbb{R}^l_{++}$ such that $(p, x) \in D$ and $p(y-x) < 0$.

Consistent choices in the sense above are then formalized by the following condition.

$$\text{For all } (p, x), (q, y) \in D : p(y-x) < 0 \text{ implies } q(y-x) \leq 0. \qquad (*)$$

Recall that a set-valued map T defined on $X \subseteq \mathbb{R}^l$ with values $T(x) \subseteq \mathbb{R}^l$ is called *quasimonotone* (in the sense of generalizing a *decreasing* real valued function of one variable) if for any $x, y \in X$ and any $x^* \in T(x), y^* \in T(y)$

$$x^* \cdot (y-x) < 0 \quad \text{implies} \quad y^* \cdot (y-x) \leq 0.$$

Thus, $(*)$ is equivalent to quasimonotonicity of the set-valued (*inverse demand*) mapping G defined by $G(x) = \left\{p \in \mathbb{R}^l_{++} | (p, x) \in D\right\}$.

Since $px = qy = 1, (*)$ can also be stated as quasimonotonicity of the set-valued *(demand)* mapping F defined by $F(p) = \left\{x \in \mathbb{R}^l_{+} | (p, x) \in D\right\}$:

$$\text{For all } x \in F(p), y \in F(q) : (p-q)y < 0 \text{ implies } (p-q)x \leq 0.$$

In Section 2 we characterize quasimonotone demand correspondences, i.e. demand relations with $F(p) \neq \emptyset$ for all $p \in \mathbb{R}^l_{++}$, as those which can be rationalized by a complete, upper continuous, monotone, and convex preference. Since the example above has shown that completeness, continuity, and monotonicity of R yields no restriction on D_R (other than $px = 1$ for $(p, x) \in D_R$), convexity is the crucial property which leads to quasimonotone demand.

In Section 3 we consider general demand relations which are rationalized by convex (and monotone) preferences. It is shown that they are characterized by a stronger consistency condition which has been introduced as *proper quasimonotonicity* by Daniilidis and Hadjisavvas (1997)[2].

2 Quasimonotone demand correspondences

As mentioned above, a demand relation can be described by a set-valued function F which assigns to every $p \in \mathbb{R}^l_{++}$ a (possibly empty) set $F(p) \subseteq \mathbb{R}^l_+$ such that $px = 1$ for all $x \in F(p)$. If $F(p) \neq \emptyset$ for every $p \in \mathbb{R}^l_{++}, F$ is called a *demand correspondence.*

Thus, F is quasimonotone if for all $p, q \in \mathbb{R}^l_{++}$ and all $x \in F(p), y \in F(q)$

$$py < 1 \quad \text{implies} \quad qx \geq 1.$$

The definition of (weak) rationalizability can be stated as follows.

F is rationalized by the preference R, if $F(p) \subseteq F_R(p)$ for all $p \in \mathbb{R}^l_{++}$, where

$$F_R(p) = \{x \in B(p) | xRy \text{ for all } y \in B(p)\}.$$

Which properties of a preference R guarantee that F_R and, consequently, any F rationalized by R is quasimonotone? An answer gives the following

Proposition 1. If R is convex and locally nonsatiated, then F_R is quasimonotone.

Proof. Assume that F_R is not quasimonotone, i.e. there are $p, q \in \mathbb{R}^l_{++}$ and $x \in F_R(p), y \in F_R(q)$ such that $py < 1$ and $qx < 1$. Since $px = qy = 1$, we obtain for $z = \frac{1}{2}x + \frac{1}{2}y$ that $pz < 1$ and $qz < 1$. Hence, there exists a neighborhood U of z with $U \subseteq B(p) \cap B(q)$. This implies xRz' and yRz' for all $z' \in U$. By convexity of R, zRz' for all $z' \in U$, i.e. R is locally satiated at z. $\square$

The converse of this proposition is also true as we will show below. Actually, we want to prove a stronger result. In order to ensure that a demand correspondence actually exists, the consumer's preference should enable him to make at least some choice for any given price vector. This is surely not necessarily the case if the preference is only known to be convex and locally nonsatiated.

Sufficient conditions for R such that F_R is a demand correspondence are given in the following

Lemma. Let R be a preference such that the following conditions are satisfied:

(i) R is *upper continuous*, i.e. $R(x)$ is closed for every $x \in \mathbb{R}^l_+$.

(ii) R has the *KKM-property,* i.e. the convex hull $\text{co}\{x_1, \ldots, x_n\}$ of finitely many elements in $\mathbb{R}^l_+$ is always contained in $\bigcup_{i=1}^n R(x_i)$.

Then $F_R(p) \neq \emptyset$ for all $p \in \mathbb{R}^l_{++}$.

Proof. By definition, $x^* \in F_R(p)$ if $x^* \in B(p)$ and $x^* R x$ for all $x \in B(p)$ which is equivalent to $x^* \in \bigcap_{x \in B(p)} R(x) \cap B(p)$. Since $B(p)$ is compact, this intersection is nonempty by Fan's Lemma (1961)[3]. □

In the presence of additional assumptions, convexity of R ensures the KKM-property for R. This is proved in

Proposition 2. A complete, upper continuous, monotone and convex preference R has the KKM-property.

Proof. Assume that the claim is not true, i.e. there are $x_1, \ldots, x_n \in \mathbb{R}^l_+$ and $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that $x = \sum_{i=1}^n \lambda_i x_i \notin R(x_i)$ for $i = 1, \ldots, n$.

By upper continuity of R, the sets $W(x_i) = \mathbb{R}^l_+ \backslash R(x_i)$ are open. Since $x \in \bigcap_{i=1}^n W(x_i)$ and this intersection is also open, there is $y \gg x$ such that $y \in \bigcap_{i=1}^n W(x_i)$. Hence, $y \in W(x_i)$ or, equivalently, $y \notin R(x_i)$ for $i = 1, \ldots, n$. Thus, completeness of R implies $x_i \in R(y)$ for $i = 1, \ldots, n$. By convexity of $R, x \in R(y)$. Since $y \gg x, R$ cannot be monotone. □

Now we characterize quasimonotone demand correspondences in terms of properties of preferences which rationalize those demands.

Theorem 1. Let F be a demand correspondence. Then the following conditions are equivalent:

(i) F is quasimonotone.

(ii) F can be rationalized by a complete, upper continuous, monotone, and convex preference R.

(iii) F can be rationalized by a locally nonsatiated and convex preference R.

Proof. (ii) trivially implies (iii) and (i) follows immediately from (iii) by Proposition 1. Thus, it remains to prove that (i) implies (ii).

Let F be a quasimonotone demand correspondence. We first define a preference S on $\mathbb{R}^l_+$ (now viewed as price space) by

$$pSq \text{ iff } \begin{cases} p \in \mathbb{R}^l_+ & \text{if } q \notin \mathbb{R}^l_{++} \\ \forall y \in F(q) : (p-q)y \geq 0 & \text{if } q \in \mathbb{R}^l_{++}. \end{cases}$$

It is straightforward to check upper continuity, monotonicity and convexity of S. In order to prove that S is complete, assume that $\neg pSq$, i.e. $q \in \mathbb{R}^l_{++}$ and there exists $y \in F(q)$ such that $(p-q)y < 0$. Since F is quasimonotone, it follows for $p \in \mathbb{R}^l_{++}$ that $(p-q)x \leq 0$ or, equivalently, that $(q-p)x \geq 0$ for all $x \in F(p)$. By definition of S, we obtain qSp, i.e. S is complete.

S is now used to define a kind of inverse ("dual") demand correspondence G_S.

Assume that $x \in \mathbb{R}^l_{++}$. Then the set $B(x) = \{p \in \mathbb{R}^l_+ | px \leq 1\}$ is a "dual" budget set and, by the Lemma and Proposition 2, the set $G_S(x) = \{p \in B(x) | pSq$ for all $q \in B(x)\}$ is nonempty. Moreover, by Proposition 1, G_S is quasimonotone.

We now use G_S to obtain a preference R on $\mathbb{R}^l_+$ (viewed as commodity space) in the same way as we derived S from F, i.e. we define

$$yRx \text{ iff } \begin{cases} y \in \mathbb{R}^l_+ & \text{if } x \notin \mathbb{R}^l_{++} \\ \forall p \in G_S(x) : p(y-x) \geq 0 & \text{if } x \in \mathbb{R}^l_{++}. \end{cases}$$

By the same arguments as above, R is complete, upper continuous, monotone, and convex. It remains to show that R rationalizes F.

For an arbitrary $q \in \mathbb{R}^l_{++}$, let $y \in F(q)$. We have to prove that yRx for all $x \in B(q)$. This is trivial for $x \notin \mathbb{R}^l_{++}$. Consider $x \in B(q), x \in \mathbb{R}^l_{++}$ and $p \in G_S(x)$. Since $q \in B(x)$ it follows by definition of G_S, that pSq. This implies, by definition of S, that $(p-q)y \geq 0$. Since $qy = px = 1$, this is equivalent to $p(y-x) \geq 0$. Thus, by definition of R, we have shown that yRx. □

3 Properly quasimonotone demand relations

The characterization of quasimonotone demand in the previous section used essentially that F is a demand correspondence. According to the spirit of revealed preference theory, F should be interpreted as a set of observed choices. Thus, one can argue that the obtained result requires too many observations since for each price vector there has to be at least one. It would be desirable to extend the characterization to arbitrary demand relations, especially including those with finitely many elements. However, Theorem 1 does not hold if only the assumption that F is a correspondence is dropped. This is shown by the following

Example. Let $D = \{(p_1, x_1), (p_2, x_2), (p_3, x_3)\}$ where $p_1 = (1, 1, 0.5), p_2 = (0.5, 1, 1), p_3 = (1, 0.5, 1)$ and $x_1 = (1, 0, 0), x_2 = (0, 1, 0), x_3 = (0, 0, 1)$.

Then $p_i x_j < 1$ only if $(i, j) \in \{(1, 3), (2, 1), (3, 2)\}$. Since in each of these cases we obtain $p_j x_i = 1, D$ is quasimonotone. However, D cannot be rationalized by a convex and monotone preference: If R rationalizes D then, since $x =$

$(0.4, 0.4, 0.4) \in B(p_i)$, it follows that $x_i R x$ for $i = 1, 2, 3$. Convexity of R would imply $(1/3, 1/3, 1/3)Rx$, i.e. R cannot be monotone.

This argument shows that a stronger property than quasimonotonicity is necessary for D to be rationalized by a convex and monotone (or locally nonsatiated) preference.

A demand relation D is called *properly quasimonotone* if there do not exist $(p_1, x_1), \dots, (p_n, x_n) \in D$ and $\lambda_1, \dots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that for $x = \sum_{i=1}^n \lambda_i x_i$ the strict inequality $p_i x < 1$ holds for $i = 1, \dots, n$.

This notion corresponds to proper quasimonotonicity of the set-valued map $G(x) = \{p \in \mathbb{R}^l_{++} | (p, x) \in D\}$ in the sense defined by Daniilides and Hadjisavvas (1997). It obviously implies quasimonotonicity: If D is not quasimonotone, there are $(p, x), (q, y) \in D$ such that $py < 1$ and $qx < 1$, hence, $p(\frac{1}{2}x + \frac{1}{2}y) < 1$ and $q(\frac{1}{2}x + \frac{1}{2}y) < 1$, i.e. D is not properly quasimonotone.

It is now straightforward to improve Proposition 1 by deriving the stronger necessary condition of proper quasimonotonicity. We even obtain the following result which is analogous to Theorem 1.

Theorem 2. Let D be a demand relation. Then the following conditions are equivalent

(i) D is properly quasimonotone.

(ii) D can be rationalized by a reflexive, upper continuous, monotone, and convex preference R.

(iii) D can be rationalized by a locally nonsatiated and convex preference R.

Proof. Since (ii) trivially implies (iii) it remains to prove the implications (iii)$\Rightarrow$(i) and (i)$\Rightarrow$(ii).

(iii)$\Rightarrow$(i): If D is not properly quasimonotone there exist $(p_1, x_1), \dots, (p_n, x_n) \in$

D such that for $x = \sum_{i=1}^n \lambda_i x_i$ with $\lambda_1, \ldots, \lambda_n > 0$ and $\sum_{i=1}^n \lambda_i = 1$ the inequality $p_i x < 1$ holds for $i = 1, \ldots, n$. Hence, there is a neighborhood U of x with $p_i y < 1$ for all $y \in U$ and every i. If R rationalizes D, this implies $x_i R y$ for all $y \in U$ and every i. Convexity of R would imply that xRy for all $y \in U$, i.e. R is locally satiated at x.

(i)⇒(ii): For any $x \in \mathbb{R}_+^l$, define the set $R_D(x)$ of all commodity bundles which are weakly revealed preferred to x by

$$R_D(x) = \{y \in \mathbb{R}_+^l | \exists (q, y) \in D : x \in B(q)\}.$$

It is obvious that R rationalizes D if and only if $R_D(x) \subseteq R(x)$ for all $x \in \mathbb{R}_+^l$. Indeed, the latter condition states that yRx whenever $x \in B(q)$ and $(q, y) \in D$ or, equivalently, that $D \subseteq D_R$.

Now we claim that proper quasimonotonicity of D implies that there do not exist $x, y \in \mathbb{R}_+^l$ such that $y \in \text{co} R_D(x)$ and $y \ll x$ (actually, we even have equivalence but only this implication is needed here.)

Assume that $y \in \text{co} R_D(x)$ and $y \ll x$. By definition, it follows that there are $(p_1, x_1), \ldots, (p_n, x_n) \in D$ and $\lambda_1, \ldots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$ such that $y = \sum_{i=1}^n \lambda_i x_i$ and $p_i x \leq 1$ for $i = 1, \ldots, n$. Since $y \ll x$, we obtain $p_i y < 1$ for $i = 1, \ldots, n$. Hence, by definition, D is not properly quasimonotone.

It is obvious that we even obtain the nonexistence of $y \in \text{co}(R_D(x) \cup \{x\})$ with $y \ll x$.

Thus, if we define a preference R by $R(x) = \text{clco}(R_D(x) \cup \{x\})$, then the sets $R(x)$ are closed, convex and contain x but no y with $y \ll x$. Thus R has the claimed properties. Since $R_D(x) \subseteq R(x)$, D is rationalized by R. □

Compared with Theorem 1, the result does not claim the existence of a complete rationalizing preference with the stated properties. It remains an open question if there is such a rationalization.

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