Evolutionary Dynamics on Infinite Strategy Spaces[*]

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Abstract

The study of evolutionary dynamics was so far mainly restricted to finite strategy spaces. In this paper we show that this unsatisfying restriction is unnecessary. We specify a simple condition under which the continuous time replicator dynamics are well defined for the case of infinite strategy spaces. Furthermore, we provide new conditions for the stability of rest points and show that even strict equilibria may be unstable. Finally, we apply this general theory to a number of applications like the Nash demand game, the War of Attrition, linear–quadratic games, the harvest preemption game, and games with mixed strategies.

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1   Introduction

The recent years have seen a surge in the studies on evolutionary game the-
ory and much progress has been made in the understanding of evolutionary
dynamics. However, most studies have focused on finite strategy sets, es-
pecially when continuous time dynamics were involved (e.g. the replicator
dynamics).[1] The present paper extends previous work for studying evolu-
tionary dynamics with arbitrary, finite or infinite, strategy sets in $\mathbb{R}^n$. The
results we obtain should allow one to study evolutionary processes in a host
of applications where this hitherto could not be done in a natural way.[2]

It is not accidental that many relevant games in economics are modelled
with a continuous strategy space, e.g. bargaining games, games of timing,
oligopoly games, public good games or all games in which players are allowed
to use mixed strategies. To study such games with an arbitrarily chosen
finite approximation seems to be an unnecessary detour given that the direct
way is available and often more convenient.

Another reason for considering infinite strategy spaces is that one is
often interested in whether the infinite case is the limit of successively finer
approximations. In this case one needs the infinite case as a reference point.

Furthermore, in some cases the continuity of the strategy set really mat-
ters. For example, Gale et al. (1995) show that non–subgame perfect equi-
libria in a finite approximation of the ultimatum game cannot be ruled out
as outcomes of an evolutionary process. But Binmore and Seymour (1995)
find that only the subgame perfect equilibrium is asymptotically stable with
respect to the noisy replicator dynamics when a continuous strategy set
is considered. Below we will present further examples where dynamics on
continuous strategy sets yield different results from the finite case.

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[1] Exceptions are Hopkins and Seymour (1996) who study the replicator dynamics but
only on spaces of probability distributions with a continuous density, Bomze (1990, 1991),
Seymour (1998), and Friedman and Yellin (1996). The latter study gradient dynamics.
The special case of mixed strategies has been studied by Akin (1982), Hines (1982), and
Zeeman (1981).

[2] An example where our approach has already fruitfully been applied (besides the ones
in Section 5) is the evolution of preferences (Huck, Kirchsteiger, and Oechssler, 1997; see
also Ely and Yilankaya, 1997).

In the evolutionary context a population is identified with the aggregate play of its members. Formally, a population is a probability measure on the strategy set. If the strategy set is infinite, the set of populations is a subset of the infinite dimensional vector space of finite signed measures. We build on work by Bomze (1990, 1991) for studying replicator dynamics on this Banach space. Bomze shows that the replicator dynamics are well defined if the *mean* payoff function of strategy x against population P is Lipschitz continuous. We show that this assumption is always satisfied in pairwise encounters if the underlying *pairwise* payoff function is bounded. In particular, no continuity assumption for payoffs is needed in order for the replicator dynamics to be well defined. This allows to study many economically interesting games with discontinuous payoff functions.

Much work in evolutionary game theory (see Weibull, 1995, for an overview) has been done on bridging the gap between static stability concepts (like ESS) and explicit dynamics (like the replicator dynamics). In the finite strategy case one important fact is that an ESS, and a fortiori, a strict equilibrium, is asymptotically stable (Hofbauer et al., 1979). A surprising result of our analysis is that in the infinite case such results cannot be taken for granted. We provide an example showing that even strict Nash equilibria may be unstable with respect to the replicator dynamics, even when the strategy space is compact and the payoff function is continuous. In general, stronger static conditions are required for dynamic stability. We prove that one such condition, namely uninvadability, is sufficient for dynamic stability of a homogeneous population state.

After having set up the general framework, we study a number of specific problems where a continuous strategy space is particularly natural. Namely, we consider the replicator dynamics in the context of the Nash demand game, the War of Attrition, linear–quadratic games, a harvest preemption game, and games with mixed strategies.

In the Nash demand game we find that the symmetric efficient equilibrium is stable and weakly attracting with respect to the replicator dynamics. In the War of Attrition the replicator dynamics converge to the unique mixed equilibrium from initial states which are within finite Kullback–Leibler dis-

tance from the equilibrium.

Linear–quadratic games are games in which the payoff is quadratic in the own action and linear in the opponent's action. This class of games includes some specifications of Common Pool Resource problems, public good games, and Cournot duopolies. We show that the replicator dynamics weakly converge to the unique equilibrium from all initial states with positive weight on the equilibrium strategy.

The harvest preemption game has the same structure as a Bertrand duopoly with homogeneous products. We show that if the strategy space is slightly restricted, then there exists a unique (and very inefficient) equilibrium which is strongly attracting with respect to the replicator dynamics. The unrestricted case, however, remains an open question.

Finally, the set of mixed strategies of a finite game can also be seen as an infinite strategy space – though one with the particularly nice structure inherent to the simplex. We show that mixed strategies also fit in our framework and review results obtained by Hines (1982) and Zeeman (1981). The advantage of using replicator dynamics on the space of mixed strategies is that – in contrast to dynamics on pure strategy spaces – evolutionary stable strategies are characterized by the asymptotically stable rest points of the replicator dynamics.

The paper is organized as follows. In the next section we describe the framework for analyzing dynamics on infinite dimensional spaces. In Section 3 we introduce the replicator dynamics and show that they are well defined for infinite strategy spaces. Section 4 is devoted to the specification of stability conditions. Section 5 contains the applications, Nash demand game, War of Attrition, linear–quadratic game, the harvest preemption game, and games with mixed strategies. Finally, Section 6 concludes. Some useful facts about the variational norm and a number of proofs are relegated to an appendix.

2 Formulation of the game

We consider a game with strategy set S. The usual approach in the literature on evolutionary dynamics is to assume a finite strategy set $S = \{1, \ldots, n\}$. Here, we generalize this by allowing for arbitrary (Borel) strategy sets $S \subset \mathbb{R}^n$. The Borel σ–algebra on S is denoted by $\mathcal{B}$.

We restrict ourselves to the case of symmetric two–player games, though the setup can be extended to the asymmetric case. Let $f : S \times S \to \mathbb{R}$ be a bounded, Borel measurable function, where $f(x, y)$ is the payoff for player 1 when she plays x and player 2 plays y.

A *population* is identified with the aggregate play of its members and is described by a probability measure P on the measure space $(S, \mathcal{B})$. The simplex of all populations is denoted by Δ. The average payoff of population P against population Q is

$$E(P, Q) = \int_S \int_S f(x, y) Q(dy) P(dx). \tag{1}$$

The aim is to study the evolution of populations over time. For obvious reasons it is more convenient to work with a vector space. Since Δ is not a vector space, we work with the linear span of Δ, that is the space $\mathcal{M}^e(S, \mathcal{B})$ of all signed measures. Recall that ν is a *signed measure* on $(S, \mathcal{B})$ if there are two finite measures μ^1 and μ^2 such that for all sets $A \in \mathcal{B}$, $\nu(A) = \mu^1(A) - \mu^2(A)$. What kind of norm is appropriate to describe the "distance" between two populations? Following Bomze (1990) we propose the supremum or variational norm.

Definition 1 *The variational norm on $\mathcal{M}^e(S, \mathcal{B})$ is given by*

$$\|\mu\| = \sup_f \left| \int f \, d\mu \right|,$$

where the sup *is taken over all measurable functions $f : S \to \mathbb{R}$ bounded by* 1, $\sup_{s \in S} |f(s)| \leq 1$.

Endowed with the variational norm, $\mathcal{M}^e$ is a Banach space (cf. Alt, 1992), that is, every Cauchy sequence in $\mathcal{M}^e$ converges to an element in

$\mathcal{M}^e$. Some useful facts about the variational norm are collected in the appendix. For example, convergence in the variational norm is equivalent to convergence in the $\mathcal{L}^1$–sense of the densities if these exist. In the finite case convergence in the variational norm is equivalent to pointwise convergence of probabilities. Thus, the case of finite strategy sets is contained in our framework as a special case.

Remark 1 The variational norm is a very strong measure of distance. However, we think that it may be a plausible norm from an evolutionary point of view. In evolutionary game theory, one frequently considers some equilibrium population P and studies its performance against some mutated population $P(\varepsilon) = (1 - \varepsilon)P + \varepsilon Q$.[3] These two populations are close in the strong sense that for all sets of strategies $A \in \mathcal{B}$, one has $|P(A) - P(\varepsilon)(A)| \leq \varepsilon$ uniformly. They are therefore close in the variational norm, as they should be; indeed, their distance is at most 2ε, compare (14). On the other hand, two homogeneous populations δ_x and δ_y have maximal distance in the variational norm for $x \neq y$, even if the strategies x and y are close in the usual metric. This might appear somehow implausible at first sight. But two distinct homogeneous societies are quite far from one another in evolutionary terms because everyone in a population has to mutate in order to convert one population into the other, which indeed would be a very rare event.

Remark 2 Although it is possible, in principle, to formulate replicator dynamics with weaker topologies, this would require stronger assumptions on the payoff function f.[4] To give an example, replicator dynamics for the weak topology are only well defined for *continuous* payoff functions. In general, the weaker the topology, the smaller is the dual space, and, thus, the class of admissible payoffs. The variational norm has therefore the advantage of allowing for the largest class of payoff functions.

Since we want to study the dynamics of populations as time unfolds, we have to deal with curves $m : \mathbb{R}_+ \rightarrow \mathcal{M}^e$. Such a function is called continuously differentiable if there exists a continuous function $m' : \mathbb{R}_+ \rightarrow$

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[3] In particular, this view is employed in the ESS and related concepts.

[4] See Oechssler and Riedel (1999) for the consequences of employing the weak topology.

$\mathcal{M}^e$ with

$$\lim_{h \to 0} \frac{m(t + h) - m(t)}{h} = m'(t) \, ,$$

where the limit has to be taken with respect to the variational norm. Let $F : \mathcal{M}^e \to \mathcal{M}^e$ be continuous. A continuously differentiable function m with

$$m'(t) = F(m(t)), \quad m(0) = \mu \tag{2}$$

is called a solution to the ordinary differential equation (2).

A crucial fact is that with Lipschitz–conditions one has always a unique solution to such initial value problems.

Theorem 1 *Suppose that F is bounded and satisfies a global Lipschitz condition:*

$$\exists K > 0 \ \text{s.t.} \ \forall \mu, \nu \in \mathcal{M}^e, \ \|F(\mu) - F(\nu)\| \leq K \, \|\mu - \nu\| \, .$$

Then, a unique solution of the ordinary differential equation (2) exists on $[0, \infty)$.

For a proof see e.g. Zeidler (1986, Corollary 3.9).

3 The Replicator Dynamics

The dynamics most widely studied in the literature on evolutionary game theory are the replicator dynamics. They formalize the idea that in a dynamic process of evolution a strategy x should increase in frequency if it is a successful strategy in the sense that individuals playing this strategy obtain a higher than average payoff. Formally, the success (or lack of success) of a strategy x if the population is Q is given by the difference

$$\sigma(x, Q) := \int_S f(x, y) Q(dy) - \int_S \int_S f(x, y) Q(dy) Q(dx) = E(\delta_x, Q) - E(Q, Q) \, .$$

The idea of replicator dynamics is that the relative increment of the frequency of a set of strategies is given exactly by the average success of strategies in that set.

Definition 2 *The ordinary differential equation*

$$Q'(t)(A) = \int_A \sigma(x, Q(t)) \, Q(t)(dx), \qquad Q(0) = P \qquad (3)$$

for all $A \in \mathcal{B}$, is called replicator dynamics.

Note that by taking $A = \{x\}$ we get the usual formulation of the replicator dynamics for the finite strategy case.

One of the main results of this paper is the following extension of work by Bomze (1991) for the present framework of pairwise encounters.[5] Bomze assumes that σ is Lipschitz continuous in P and shows then that the replicator dynamics are well defined. In the following theorem we prove that for pairwise encounters this requirement reduces to f being bounded.

Theorem 2 *If the payoff function f is bounded, then the replicator dynamics are well defined.*

The intuition for Theorem 2 is the following: when the payoff function f is bounded, the bilinear functional $E(P, Q)$ given by (1) is *continuous* in the variational topology. This implies that the right-hand side of the replicator dynamics, $\int_A \sigma(x, Q(t)) \, Q(t)(dx)$, is a Lipschitz continuous function on Δ. A solution to the ODE thus exists and it remains to show that the trajectories $(Q(t))$ of any solution never leave the set of populations Δ. This is done in the appendix.

4 Stability concepts

We start with the classical concept of evolutionary stability introduced by Maynard Smith (1974).

Definition 3 *A population P is called an* evolutionary stable state *(ESS) if for every "mutation" Q, there is an invasion barrier $\varepsilon(Q) > 0$ such that for all $0 < \eta \leq \varepsilon$*

$$E(P, (1 - \eta)P + \eta Q) > E(Q, (1 - \eta)P + \eta Q). \qquad (4)$$

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[5] See also the recent paper by Seymour (1998), who develops a similar framework to ours for the asymmetric case.

As we will see below the concept of ESS is often too weak in the context of infinite strategy spaces. The following stronger notion was suggested by Vickers and Cannings (1987).[6]

Definition 4 *A population P is called* uninvadable *if there is a uniform invasion barrier, that is, an $\varepsilon > 0$ such that (4) holds for all Q and all $0 < \eta \leq \varepsilon$.*

Uninvadability requires a uniform invasion barrier for all possible mutations Q. In general, one could require even more. Up to now, we have considered the case in which a small fraction η of the populations changes *arbitrarily*. More generally, one could look at the case where the whole population is allowed to shift, but only in a manner that the *distance* between the original and the mutated population remains small. This yields the following definition introduced by Bomze (1990).

Definition 5 *A population P is called* strongly uninvadable *if there is a barrier $\varepsilon > 0$ such that for all populations $R \neq P$ with $\|R - P\| \leq \varepsilon$, we have*

$$E(P, R) > E(R, R).$$

We collect some useful facts about the various stability concepts. The first and third point are known, the second fact is new. The proof of the others is given for completeness.

Lemma 1   *1. Every strongly uninvadable population is uninvadable, and every uninvadable population is evolutionary stable.*

2. If a discrete measure $P = \sum_{j=1}^{n} p_j \delta_{x_j}$ is uninvadable, then it is also strongly uninvadable.

3. If the strategy space S is finite, then the three concepts coincide.

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[6]For an introduction to static stability concepts with infinite strategy spaces see also Bomze and Pötscher (1989).

Proof. The first statement is obvious. For the second, let $P = \sum_{j=1}^{n} p_j \delta_{x_j}$ be uninvadable, and assume without loss of generality that $p_j > 0$ for all j. Let ε be a uniform invasion barrier and set $\tilde{\varepsilon} := \varepsilon \min p_j$. Now assume $\|R - P\| \leq \tilde{\varepsilon}$. Set $r_j := R(\{x_j\})$. Then $\eta := \max |p_j - r_j| / p_j \leq \varepsilon$. Define a measure Q via $Q = \frac{1}{\eta}(R - (1-\eta)P)$. Then it is easy to check that $Q \in \Delta$. Hence, R can be written as $R = (1-\eta)P + \eta Q$ for some $\eta \leq \varepsilon$ and some population Q. Since P is uninvadable, it follows that $E(P, R) > E(Q, R)$ and hence $E(P, R) > E(R, R)$.

That every ESS is uninvadable in the finite case is well known (see e.g. Vickers and Cannings, 1987). Since with a finite strategy space all populations are discrete measures, the second statement implies that every uninvadable population is also strongly uninvadable. ∎

The next definition specifies the *dynamic* stability concepts we will use in the following.

Definition 6 *Let $Q^\star$ be a rest point of the replicator dynamics,*

$$\sigma(\cdot, Q^\star) = 0 \quad Q^\star - a.e.$$

Then

- *$Q^\star$ is called Lyapunov stable if for all $\varepsilon > 0$ there exists an $\eta > 0$ such that $\|Q(0) - Q^\star\| < \eta \Rightarrow \|Q(t) - Q^\star\| < \varepsilon$ for all $t > 0$.*

- *$Q^\star$ is called strongly attracting if there exists $\varepsilon > 0$ such that $\|Q(0) - Q^\star\| < \varepsilon \Rightarrow \|Q(t) - Q^\star\| \rightarrow 0$.*

- *$Q^\star$ is called weakly attracting if there exists $\varepsilon > 0$ such that $\|Q(0) - Q^\star\| < \varepsilon \Rightarrow Q(t) \rightarrow Q^*$ in distribution.*

In the finite case the last two concepts coincide. Together with Lyapunov stability they are called asymptotic stability. As is well known in the finite case an ESS is sufficient for asymptotic stability (Hofbauer et al., 1979). A fortiori, all strict equilibria are asymptotically stable. One may wonder whether this result survives in the infinite case. Somewhat surprisingly it

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does not as the following example shows. In fact, strict equilibria need not even be Lyapunov stable.[7]

Example 1 *Consider a game with compact strategy set $S = [-1, 1]$ and differentiable payoff function*

$$f(x, y) = -x^4 + 4xy.$$

It can easily be checked that $(0, 0)$ is a strict Nash equilibrium. However, it is not uninvadable and it is not Lyapunov stable. For all $\varepsilon > 0$, there exist strategies $x \in S$ such that

$$\sigma(x, (1 - \varepsilon)\delta_0 + \varepsilon\delta_x) = 4\varepsilon x^2 - x^4 > 0. \tag{5}$$

Therefore, δ_0 is not uninvadable. Moreover, the replicator dynamics imply that for $Q(0) = (1 - \varepsilon)\delta_0 + \varepsilon\delta_x$, with some x such that (5) holds, the weight on x, $\eta(t) := Q(\{x\})$, increases according to

$$\frac{\eta'(t)}{\eta(t)} = \sigma(x, Q(t)) = 4\eta(t)x^2 - x^4 \geq 4\eta(0)x^2 - x^4 > 0.$$

Thus, $\eta(t)$ increases to 1 and δ_0 is not Lyapunov stable.

This example shows that stability results which are taken for granted in the finite strategy case need not apply in the infinite case.[8] But rather than discrediting the infinite case, we think that the example throws a critical light on the stability concepts in the finite case. While it is true that the strict equilibrium $(0, 0)$ is asymptotically stable if the game is played with any finite grid size, its basin of attraction vanishes as the grid becomes finer. It seems that there are some strict equilibria which are less robust than others, in particular, those which do not converge to a stable equilibrium as

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[7] There exist examples in the literature showing that strict equilibria need not be un-invadable. However, as far as we know they relied on either non–compactness of S or on discontinuities of f (see e.g. Vickers and Cannings, 1987).

[8] It should be pointed out that the example is not an artefact of our chosen topology. In Oechssler and Riedel (1999) we show that with the weak topology similar examples can easily be constructed.

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the grid size goes to zero. In the infinite case stronger concepts than ESS are necessary. For this we have the following useful result, which connects the static and the dynamic stability concepts.

Theorem 3 *If $Q^\star = \delta_x$ is an uninvadable, homogeneous population, then*

- $Q^\star$ is Lyapunov stable;

- if additionally the payoff function f is continuous, then $Q^\star$ is weakly attracting.

The second part of the Theorem is an adaptation of Theorem 2 in Bomze (1990) to our setting.

Proof. Since δ_x is uninvadable (and therefore strongly uninvadable by Lemma 1), there exists an $\varepsilon > 0$ such that for all $R \in \Delta$ with $\|R - \delta_x\| \leq \varepsilon$

$$E(\delta_x, R) > E(R, R).$$

For $Q(0) = \delta_x$ there is nothing to show. Assume $Q(0) \neq \delta_x$. The function $H(t) := Q(t)(\{x\})$ is continuously differentiable. By Theorem 6, Appendix, we have

$$H(t) = H(0) \exp\left(\int_0^t \sigma(x, Q(s))ds\right), \tag{6}$$

hence

$$\frac{H'(t)}{H(t)} = \sigma(x, Q(t)).$$

We claim that H is strictly increasing. By assumption, $H'(0) = \sigma(x, Q(0)) > 0$. Suppose that H' eventually became zero and set

$$t_0 := \inf\left\{t \geq 0; H'(t) = 0\right\}.$$

Then $H'(t_0) = 0$ since the set $\{t \geq 0; H'(t) = 0\}$ is closed. For all $s < t_0$ we have $H'(s) > 0$ which implies $H(s) > H(0)$, hence $\|Q(s) - \delta_x\| < \varepsilon$. Continuity of the trajectory yields $\|Q(t_0) - \delta_x\| \leq \varepsilon$, hence $H'(t_0) = \sigma(x, Q(t_0)) > 0$ by uninvadability, a contradiction. Therefore, H is strictly increasing which implies that $\|Q(t) - \delta_x\| < \varepsilon$.

We show next that the fitness differential $\sigma(x, Q(t))$ vanishes. By (6), the convergence of $Q(t)(\{x\}) = H(t)$ implies $\int_0^\infty \sigma(x, Q(s))ds < \infty$. The claim follows if we show the equicontinuity of the map $t \mapsto \sigma(x, Q(t))$. Since the conditions of Lemma 3, Appendix, are satisfied, we have by (15),

$$|\sigma(x, Q(t)) - \sigma(x, Q(s))| \le L \|Q(t) - Q(s)\| .$$

The replicator dynamics and the boundedness of σ on Δ, see (16), yield for every set A

$$|Q(t)(A) - Q(s)(A)| = \left| \int_s^t Q'(u)(A)du \right|$$
$$= \left| \int_s^t \int_A \sigma(\xi, Q(u))Q(u)(d\xi)du \right| \le \sigma_\infty |t - s| .$$

Hence, using (14),

$$|\sigma(x, Q(t)) - \sigma(x, Q(s))| \le 2L\sigma_\infty \|t - s\| ,$$

which implies Lipschitz and hence equicontinuity of the map $t \mapsto \sigma(x, Q(t))$.

If the payoff function f is continuous, the expected payoff difference $E(\delta_x, Q) - E(Q, Q)$ is continuous in Q with respect to the weak topology. The set Δ of all populations is compact in the weak topology. Let P be a weak accumulation point of the trajectory $(Q(t))$. By the preceding, $0 = \lim \sigma(x, Q(t)) = \lim [E(\delta_x, Q(t)) - E(Q(t), Q(t))] = E(\delta_x, P) - E(P, P)$. By the first part of the theorem, δ_x is stable, hence P is close to δ_x. By uninvadability, $P = \delta_x$. ∎

The proof of the preceding theorem shows that the replicator dynamics increase the weight on the pure strategy x if δ_x is uninvadable. Therefore, the weight $Q(t)(\{x\})$ converges and the growth rate of the strategy must vanish. We state this useful fact as a corollary.

Corollary 1 *If $Q^\star = \delta_x$ is an uninvadable, homogeneous population, then the fitness differential vanishes:*

$$\sigma(x, Q(t)) \to 0.$$

5 Applications

For many games it is more natural to think of strategies as belonging to a continuum, in particular, if strategies involve the timing of actions or the choice of prices or locations. Even if smallest measurement units for quantities or prices exist, it is often more convenient to model them as continuous. In this section we present a number of examples where the replicator dynamics are applied to infinite strategy spaces.

5.1 Nash demand game

One simple example is the Nash demand game. Two players have to decide how to divide a resource of size 1. Both players simultaneously submit demands, x and y (i.e. $S = \mathbb{R}_+$). If the demands are feasible, both get what they demanded. If not, both receive nothing. Thus the payoff function is

$$f(x,y) = \left\{ \begin{array}{ll} x & \text{if } x+y \leq 1 \\ 0 & \text{if } x+y > 1 \end{array} \right. .$$

The unique efficient symmetric equilibrium of this game is $(\frac{1}{2}, \frac{1}{2})$.[9] Since this equilibrium is strict, it is an ESS. The next proposition shows that it is uninvadable and weakly attracting.

Proposition 1 *In the Nash demand game the homogenous population $\delta_{1/2}$ is uninvadable, Lyapunov stable, and weakly attracting.*

Proof. To establish uninvadability, we need to show that $E(P,P) < E(\delta_{1/2}, P)$, for all $P \neq \delta_{1/2}$ with $\left\| \delta_{1/2} - P \right\| \leq \varepsilon$. All P in the ε–neighborhood of $\delta_{1/2}$ can be written as

$$P = \alpha Q^- + \beta Q^+ + (1 - \alpha - \beta)\delta_{1/2}, \tag{7}$$

where Q^- and Q^+ are some probability measures with $Q^- \left([\frac{1}{2}, 1] \right) = 0$ and $Q^+ \left([0, \frac{1}{2}] \right) = 0$, and $\alpha + \beta \leq \varepsilon$. For probability measures R, R' concentrated on $\left[0, \frac{1}{2} \right]$, one has $E(R, R') = m(R) \leq 1/2$, where $m(R) = \int x R(dx)$ denotes

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[9]Note that the asymmetric strict equilibria cannot be restpoints of one–population dynamics.

the mean value of population R. If $\beta = 0$, one has that $E(P, P) = m(P) < \frac{1}{2} = E(\delta_{1/2}, P)$.

Next consider $\beta > 0$. In general, $E(\delta_{1/2}, P) = \frac{1}{2}(1 - \beta)$. Note that $E(Q^+, P) = \alpha E(Q^+, Q^-) \leq \alpha m(Q^+) \leq \alpha$. The bilinearity of E yields

$$\sigma\left(\tfrac{1}{2}, P\right) = \tfrac{1}{2}(1 - \beta) - E(P, P)$$

$$= \tfrac{1}{2}(1 - \beta) - \alpha E(Q^-, P) - (1 - \alpha - \beta)E(\delta_{1/2}, P) - \beta E(Q^+, P)$$

$$\geq \tfrac{1}{2}(1 - \beta) - \alpha m\left(Q^-\right) - \tfrac{1}{2}(1 - \alpha - \beta)(1 - \beta) - \alpha\beta m\left(Q^+\right) \qquad (8)$$

$$\geq \tfrac{1}{2}(1 - \beta) - \tfrac{1}{2}\alpha - \tfrac{1}{2}(1 - \alpha - \beta)(1 - \beta) - \alpha\beta$$

$$= \tfrac{1}{2}\beta\left(1 - \beta - 3\alpha\right). \qquad (9)$$

Thus, $\sigma(1/2, P) > 0$ if $\beta > 0$ and $3\alpha + \beta < 1$, which is satisfied for ε small. Hence $\delta_{1/2}$ is uninvadable. It follows from Theorem 3 that $\delta_{1/2}$ is Lyapunov stable.

By Corollary 1 the fitness difference $\sigma\left(\tfrac{1}{2}, Q(t)\right)$ vanishes. Defining $\beta(t), \alpha(t)$ for $Q(t)$ as in equation (7), we obtain by (9) that $\beta(t) \to 0$. (8) implies then that $\alpha(t) \to 0$ or $m(Q(t)^-) \to \tfrac{1}{2}$. In both cases, $Q(t)^-$ converges in $\mathcal{L}^1$, hence weakly, to $\delta_{1/2}$. Therefore, the $\delta_{1/2}$ is also weakly attracting. ∎

5.2 The War of Attrition

An example for a game in which timing is the relevant choice is the well known War of Attrition, which has important applications in economics and biology. Consider two players fighting for a prize worth V to both players. A strategy is to choose a length of time $x \in [0, M]$ for which one is prepared to stay in the race. Fighting is costly. The payoffs are given as follows

$$f(x, y) = \begin{cases} V - y & \text{if } x > y \\ \frac{V}{2} - x & \text{if } x = y \\ -x & \text{if } x < y \end{cases}$$

that is, a player gets the prize if he stay longer in the race than his rival but has to share if they stay equally long. We assume that $M > V/2$. Otherwise waiting until the end is always profitable.

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It is obvious that no pure strategy Nash equilibrium exists. But as shown by Bishop and Cannings (1978) there is a unique, completely mixed Nash equilibrium, which has the following equilibrium distribution $P^\star$. Let $t^\star = M - V/2$.

$$P^\star([0, x]) = \begin{cases} 1 - e^{-x/V} & \text{if } x \leq t^\star \\ 1 - e^{-t^\star/V} & \text{if } t^\star < x < M \\ 1 & \text{if } x = M \end{cases}$$

Bishop and Cannings (1978) show that $P^\star$ is an ESS. In fact, they show (1978, p. 118) that the fitness differential between the equilibrium distribution $P^\star$ and any mutation Q is given by the square of the $\mathcal{L}^2$-distance of the corresponding distribution functions:

$$E(P^\star, Q) - E(Q, Q) = \left(\|P^\star - Q\|_2\right)^2 \tag{10}$$

where

$$\|\mu\|_2 := \left(\int_0^M \mu\left([s, M]\right)^2 ds\right)^{\frac{1}{2}}$$

denotes the $\mathcal{L}^2$-norm on the space of distribution functions. Since the right hand side of (10) is strictly positive for all $Q \neq P^\star$, $P^\star$ is strongly uninvadable.

In light of (10), it seems natural to use the $\mathcal{L}^2$-topology in dynamic considerations. On the space of populations Δ, this topology is equivalent to the weak topology as shown by Hindy, Huang, and Kreps (1992). The following theorem demonstrates that the replicator dynamics converge globally to $P^\star$ from all initial states $Q(0)$ which have finite Kullback–Leibler distance (or cross–entropy) with respect to $P^\star$.

Theorem 4 *Assume $Q(0)$ dominates $P^\star$ and*

$$\int \log \frac{dP^\star}{dQ(0)} dP^\star < \infty \,. \tag{11}$$

Then the replicator dynamics with initial condition $Q(0)$ converge weakly to the equilibrium distribution $P^\star$.

Proof. Because of Theorem 6, $Q(t)$ and $Q(0)$ are uniformly equivalent. Thus, the Kullback–Leibler distance

$$K(Q(t), P^\star) := \int \log \frac{dP^\star}{dQ(t)} dP^\star$$

is well defined, finite, and nonnegative. With the use of the representation for the density of $Q(t)$ with respect to $Q(0)$ obtained in Theorem 6 it follows that

$$
\begin{aligned}
0 \quad &\leq \quad K(Q(t), P^\star) \\
&= \quad \int \log \left(\frac{dQ(0)}{dQ(t)} \frac{dP^\star}{dQ(0)} \right) dP^\star \\
&= \quad K(Q(0), P^\star) - \int \int_0^t \sigma(x, Q(s)) ds \, dP^\star \\
&= \quad K(Q(0), P^\star) - \int_0^t \left(E(P^\star, Q(s)) - E(Q(s), Q(s)) \right) ds .
\end{aligned}
$$

(10) yields

$$0 \leq K(Q(0), P^\star) - \int_0^t \left(\|P^\star - Q(s)\|_2 \right)^2 ds .$$

Hence, the integral

$$\int_0^\infty \left(\|P^\star - Q(s)\|_2 \right)^2 ds < \infty \tag{12}$$

exists. It follows that the distance $\|P^\star - Q(s)\|_2$ tends to zero as $s \to \infty$ since the map $s \mapsto \|P^\star - Q(s)\|_2$ is equicontinuous, which can be seen as follows. By the triangular inequality, $\left| \|P^\star - Q(s)\|_2 - \|P^\star - Q(t)\|_2 \right| \leq \|Q(s) - Q(t)\|_2$. The $\mathcal{L}^2$-norm is dominated by the variational norm,

$$
\begin{aligned}
\|Q(s) - Q(t)\|_2 &= \left(\int_0^M \left(Q(s)\left([x, \infty)\right) - Q(t)\left([x, \infty)\right) \right)^2 dx \right)^{\frac{1}{2}} \\
&\leq \frac{M^{\frac{1}{2}}}{2} \|Q(s) - Q(t)\| ,
\end{aligned}
$$

and the trajectory $(Q(t))$ is Lipschitz, $\|Q(s) - Q(t)\| \leq 2\sigma_\infty |s - t|$, compare the proof of Corollary 1. ∎

5.3   Linear–quadratic games

In this section we consider games in which the payoff is quadratic in the own action and linear in the opponent's action. Let $S = [0, M]$, for some M large enough, be the strategy set for both players. The payoff function is given by

$$f(x, y) = ax^2 + bxy + cx + dy,$$

with $a, b < 0$ and $c > 0$. Given this assumptions there exists a unique symmetric and interior Nash equilibrium in which both players choose actions $x^* = -\frac{c}{2a+b}$. This class of games includes Common Pool Resource problems and Cournot duopolies with linear demand and quadratic or linear costs (for $d = 0$). Further, for $d > 0$ the payoffs represent that of a public good problem in which the contributions are strategic substitutes.

To study the stability properties of x^* we need the following useful fact, which is due to the model's linearity in y.

Lemma 2 $E(\delta_{x^*}, Q) > E(Q, Q)$ for all $Q \neq \delta_{x^*}$.

Proof. Let $\bar{x} := \int_0^M x Q(dx)$ denote the average action. The Lemma follows from the following chain of (in)equalities.

$$E(\delta_{x^*}, Q) = f(x^*, \bar{x}) > f(\bar{x}, \bar{x}) \geq E(Q, \delta_{\bar{x}}) = E(Q, Q).$$

The first equality follows from the linearity of the payoff function $f(x, y)$ in y. To verify the first inequality suppose $\bar{x} < x^*$ and let $r(y) := \arg\max_x f(x, y)$ denote a player's best reply. Since $\frac{\partial^2 f(x, y)}{\partial x \partial y} < 0$, $r(y)' < 0$. Thus, $r(\bar{x}) > r(x^*) = x^*$. By definition of $r(y)$ and $\bar{x} < x^*$

$$f(r(\bar{x}), \bar{x}) > f(\bar{x}, \bar{x}).$$

Concavity of $f(x, y)$ in x implies that

$$f(x^*, \bar{x}) > f(\bar{x}, \bar{x}).$$

A similar argument holds for $\bar{x} > x^*$.

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The second inequality follows directly from concavity of $f(x, y)$ in x and Jensen's inequality.

$$f(\bar{x}, \bar{x}) \geq \int_0^M f(x, \bar{x}) Q(dx) = E(Q, \delta_{\bar{x}}).$$

Finally, the last equality follows again from linearity of $f(x, y)$ in y. ∎

Lemma 2 implies in particular that δ_{x^*} is strongly uninvadable. Theorem 3, therefore, yields that the equilibrium is Lyapunov stable. We show next that replicator dynamics converge globally to the equilibrium from all initial states which put positive weight on the equilibrium.

Proposition 2 *The replicator dynamics converge weakly to the equilibrium distribution δ_{x^*} of the linear–quadratic game from any initial state with $Q(0)(\{x^*\}) > 0$. In particular, δ_{x^*} is weakly attracting.*

Proof. Since the profit difference $\sigma(x^*, Q) > 0$ for all populations Q, the weight on x^* increases with time for every initial state $Q(0)$, which puts positive probability on x^*. By Corollary 1 the fitness differential $\sigma(x^*, Q(t))$ vanishes. The mean payoff is

$$
\begin{aligned}
E(Q, Q) &= \int_0^M f(x, \bar{x}) Q(dx) \\
&= \int_0^M \left[ax^2 + bx\bar{x} + cx + d\bar{x} \right] Q(dx) \\
&= f(\bar{x}, \bar{x}) + aVar(Q).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sigma(x^*, Q) &= E(\delta_{x^*}, Q) - E(Q, Q) \\
&= f(x^*, \bar{x}) - f(\bar{x}, \bar{x}) - aVar(Q).
\end{aligned}
$$

Since $f(x^*, \bar{x}) > f(\bar{x}, \bar{x})$ by the proof of Lemma 2, $\sigma(x^*, Q(t)) \to 0$ implies that $Var(Q(t)) \to 0$ and $\bar{x} \to x^*$ (recall that $a < 0$). Thus, $Q(t) \to \delta_{x^*}$ in $\mathcal{L}^2$, which implies weak convergence. ∎

5.4 Harvest preemption game

Consider a mushroom, or some other plant for that matter, that grows in the forest. The mushroom's value to gatherers is determined by a time dependent, continuous, and bounded function $g(x) \geq 0$, where we assume that $g(x) > 0$ if and only if $x \in (0, \bar{x})$. That is, there is a date $\bar{x}$ after which the mushroom is spoiled.

Two gatherers have to decide on the time for harvesting the mushroom. While both know that it would be better to let the mushroom grow to its optimal size, both try to preempt the other in order to have the mushroom to themselves. Thus, the strategy in the harvest preemption game is timing, $x, y \in \mathbb{R}_+$, and the payoff function is given by

$$f(x, y) = \begin{cases} g(x) & x < y \\ g(x)/2 & x = y \\ 0 & x > y, \end{cases}$$

where we assume that both players get half of the mushroom if they arrive at the same time. Note, that incidentally the game resembles exactly a Bertrand duopoly with homogenous products and zero marginal cost.

It is easy to check by the usual undercutting argument that the unique Nash equilibrium of this game is for both players to choose $x = y = 0$. Note, however, that the game does not possess an ESS. In particular, δ_0 is not an ESS since

$$E(\delta_0, (1 - \varepsilon)\delta_0 + \varepsilon Q) < E(Q, (1 - \varepsilon)\delta_0 + \varepsilon Q)$$

for all Q with $Q((0, \bar{x})) = 1$. For the same reason, δ_0 is not Lyapunov stable.

For a discrete strategy space Hehenkamp (1997) demonstrates that the smallest grid point above 0 is globally stable. We can derive a similar result if we exclude from the strategy space some small open interval $(0, b)$, $b > 0$, for example because there is a minimum reaction time before one can realize that the mushroom is out of the ground. The outcome for the unrestricted strategy set, however, is still an open question.

Proposition 3 *Let $S = [b, \infty)$ for some b with $\bar{x} > b > 0$. Then δ_b is uninvadable, and, therefore, Lyapunov stable with respect to the replicator dynamics. Moreover, δ_b is strongly attracting.*

Proof. Let K be an upper bound for the payoffs. Choose

$$\eta < \left(\frac{2K}{g(b)} - 1\right)^{-1}. \tag{13}$$

We show first that $\sigma(b, R) > 0$ for every population $R \neq \delta_b$ with $\|R - \delta_b\| \leq \eta$. Every such R can be written as $(1 - \beta)\delta_b + \beta P^+$ with $P^+(\{b\}) = 0$ and $0 < \beta \leq \eta$. Then $E(\delta_b, R) = (1 + \beta)\, g(b)/2$ and

$$
\begin{aligned}
E(R, R) &= (1 - \beta)E(\delta_b, R) + \beta^2 E(P^+, P^+) \\
&\leq (1 - \beta)E(\delta_b, R) + \beta^2 K.
\end{aligned}
$$

Hence,

$$\sigma(b, R) \geq \beta\,(1 + \beta)\,\frac{g(b)}{2} - \beta^2 K > 0$$

by (13).Thus, δ_b is uninvadable and, by Theorem 3, Lyapunov stable. Now assume $\delta_b \neq Q(0)\,(\{b\}) \geq 1 - \eta$. Define $\beta(t)$ via

$$Q(t) = (1 - \beta(t))\,\delta_b + \beta(t)P^+(t)$$

as above. Then, exactly as before,

$$\sigma(b, Q(t)) \geq \beta(t)\left((1 - \eta)\,\frac{g(b)}{2} - \eta K\right) > 0.$$

By Corollary 1, $\beta(t)$ must go to zero. ∎

5.5 Mixed strategies

Replicator dynamics are usually defined on the (finite) space of pure strategies. In contrast, the static analog of an evolutionary stable strategy (ESS) allows for individuals playing mixed strategies. This is the reason why the set of ESS cannot be characterized by the stable rest points of the replicator dynamics. Every ESS is an asymptotically stable state of the finite replicator dynamics but not vice versa (see e.g. Weibull, 1995). The divergence of these concepts disappears if one defines replicator dynamics on the set of mixed strategies.

Let $< U, T >$ denote the underlying symmetric 2–player normal form game, where $U : T \times T \to \mathbb{R}$ denotes the payoff matrix and T is the finite set of pure strategies. Let n be the number of pure strategies. Let $\Delta(T) := \{ x \in \mathbb{R}^n \, | \, x_i \geq 0, \sum_{i=1}^n x_i = 1 \}$ denote the $n - 1$ dimensional simplex.

Since we are concerned with the evolution of mixed strategies, our strategy set S is given by $\Delta(T)$. The corresponding payoff function is simply

$$f(x, y) = xUy = \sum_{i=1}^n \sum_{j=1}^n x_i y_j U_{ij}.$$

Note that a mix over mixed strategies induces just another mixed strategy. Let

$$\mu := \int_{\Delta(T)} x Q(dx)$$

denote the mean mixed strategy in the population. Compare now the fitness of some subpopulation in which everyone mixes half–half between two pure strategies i and j with some other subpopulation in which half of the players choose i and the others j. Since their mean mixed strategy is the same, one cannot expect evolution to select among those subpopulations. At best one can hope that the mean mixed strategies possess some kind of stability. Such a result was proved by Hines (1982). By Lemma 5 of Zeeman (1981), the evolution of the mean strategy satisfies

$$\mu'(t) = C(Q(t))U\mu(t),$$

where $C(Q) = \int (x - \mu)(x - \mu)Q(dx)$ denotes the covariance matrix.

Proposition 4 *(Hines, 1982) Let $x^* >> 0$ be a completely mixed ESS. Then μ^* is asymptotically stable if and only if $\mu^* = x^*$.*

6   Conclusion

We have shown in this paper that the replicator dynamics can be applied to continuous strategy spaces without modification. The only condition is that the underlying payoff function must be bounded, which can often be achieved by imposing arbitrarily large bounds on the strategy space.

In games in which a continuous strategy space is more natural, e.g. when quantities, prices, timing etc. are strategies, it should be possible now to use replicator dynamics directly on the continuous strategy space rather than on a more or less suitable discretization.

We have applied the theory to a number of examples. In the Nash demand game, the War of Attrition, and linear–quadratic games, the results for the discretization are (roughly) reproduced. However, in the harvest preemption game the results of the discrete model turned out to be somewhat misleading. While in the discrete model the smallest grid point above 0 is asymptotically stable, in the continuous model with unconstrained strategy set no such stable outcome seems to exist. The shape of the limit distribution is still an open question.

A Appendix

A.1 The variational norm

It is useful to have some tools at hand that make calculations of the variational norm easier and lead at the same time to a better understanding of the induced topology. Let $P, Q \in \Delta$ denote probability measures. For probability measures we have that (cf. Shiryaev, 1995, p. 360)

$$\|P - Q\| = 2 \sup_{A \in \mathcal{B}} |P(A) - Q(A)| \,. \tag{14}$$

Thus, the maximum distance between two probability measures is 2 and is reached when the measures are orthogonal,

$$P \perp Q \Rightarrow \|P - Q\| = 2 \,.$$

To see this, take a set A with $P(A) = 1$ and $Q(A^c) = 1$ and let $f = 1_A - 1_{A^c}$, where 1_A denotes the indicator function. Then

$$\int f d(P - Q) = P(A) + Q(A^c) = 2 \,.$$

The following theorem is an important auxiliary result for our further analysis as it provides a method to calculate the variational norm if we

have densities. Let $\mu = aP - bQ$, for some nonnegative $a, b \geq 0$, be a signed measure. Assume that there is a third probability measure R that dominates[10] P, Q, hence also μ. Then the Radon–Nikodym densities $\phi = \frac{dP}{dR}$, $\psi = \frac{dQ}{dR}$ and $\xi = \frac{d\mu}{dR} = a\phi - b\psi$ exist. We have

Theorem 5 *The variational norm of μ is given by*

$$\|\mu\| = \int_S |\xi| \, dR \,.$$

In particular, the distance between probability measures is given by

$$\|P - Q\| = \int_S |\phi - \psi| \, dR \,.$$

Proof. It suffices to prove the first formula because the second formula follows by taking $a = b = 1$.

For a measurable function f bounded by 1, one has by the monotonicity of the integral with respect to R

$$\left| \int f \, d\mu \right| = \left| \int f \xi \, dR \right| \leq \int |f\xi| \, dR \leq \int |\xi| \, dR,$$

hence $\|\mu\| \leq \int |\xi| \, dR$. To show equality, set $A = \{\xi > 0\}$ and $f = 1_A - 1_{A^c}$. Then f is bounded by 1, hence

$$\|\mu\| \geq \left| \int f \, d\mu \right| = \left| \int_A \xi \, dR - \int_{A^c} \xi \, dR \right| = \int |\xi| \, dR \,,$$

because $\xi = |\xi|$ on A and $-\xi = |\xi|$ on A^c. ∎

The preceding theorem implies that the convergence of $P^n \to P$ in the variational norm is equivalent to the convergence of the densities $\frac{dP^n}{dR} \to \frac{dP}{dR}$ in the $\mathcal{L}^1$–sense with respect to the dominating measure R if the sequence (P^n) and the measure P are dominated by R.

We state this as a corollary for the case of the Lebesgue measure.

Corollary 2 *Let P^n and P have densities $v^n(x)$ and $v(x)$ with respect to the Lebesgue measure. Then*

$$P^n \to P \Leftrightarrow \int_{-\infty}^{\infty} |v^n(x) - v(x)| \, dx \to 0 \,.$$

---

[10] Every set A with $R(A) = 0$ has $P(A) = 0$.

Finally, let us have a look at the classical discrete case.

Corollary 3 *Let $S = \{1, \ldots, n\}$. Set $p_i^n = P^n(\{i\})$ and $p_i = P(\{i\})$. Then*

$$P^n \to P \Leftrightarrow p_i^n \to p_i \,, \forall i$$

Proof. The discrete measures P^n and P are dominated by the counting measure $\zeta = \sum_{k=1}^{n} \delta_k$. The densities are

$$\frac{dP^n}{d\zeta}(i) = p_i^n \,.$$

By Theorem 5,

$$\|P^n - P\| = E^\zeta \left| \frac{dP^n}{d\zeta} - \frac{dP}{d\zeta} \right| = \sum_i |p_i^n - p_i| \,.$$

The left–hand side goes to zero if and only if for every i the probabilities p_i^n tend to p^i. ∎

As we see, in the discrete case, the topology induced by the variational norm is equivalent to the pointwise convergence of probabilities.

A.2 Replicator Dynamics are well defined

The strategy for proving Theorem 2 is the following. Denote by $F(Q) = \int \sigma(x, Q) Q(dx)$ the right–hand side of the replicator dynamics. Since F is neither bounded nor globally Lipschitz continuous on $\mathcal{M}^e$, we construct in the following two lemmata an auxiliary function $\tilde{F}$ which has these properties and coincides with F on Δ. Theorem 1 then implies that the ordinary differential equation

$$Q'(t) = \tilde{F}(Q(t)), \; Q(0) = P$$

has a unique solution $(Q(t))$. Finally, in Theorem 6 we show that $Q(t)$ never leaves Δ, which implies that $(Q(t))$ also solves the replicator equation.

Lemma 3 *Suppose the following Lipschitz and boundedness conditions hold for σ*

$$\|Q\|, \|R\| \leq 2 \; \Rightarrow \; \sup_x |\sigma(x, Q) - \sigma(x, R)| \leq L \|Q - R\| \qquad (15)$$

$$\sup_{Q:\|Q\|\leq 2} |\sigma(x, Q)| \leq \sigma_\infty \, , \qquad (16)$$

where L and σ_∞ are some constants with L, $\sigma_\infty < \infty$. Then there exists a bounded, Lipschitz continuous function $\tilde{F} : \mathcal{M}^e \to \mathcal{M}^e$, which coincides with F on Δ,

$$\tilde{F}(P) = F(P), \, \forall P \in \Delta.$$

Proof. As a candidate for $\tilde{F}$, we propose

$$\tilde{F}(Q) = (2 - \|Q\|)^+ \, F(Q) \, .$$

$\tilde{F}$ is zero for $\|Q\| \geq 2$, bounded and coincides with F on Δ because probability measures have norm 1. Let Q and R be measures with $\|Q\|, \|R\| \leq 2$. Choose $P = (|Q| + |R|)/2$ as the dominating measure. Here, we use the notation $|\mu| := \mu^+ + \mu^-$, where the nonnegative measures μ^+ and μ^- form the Jordan decomposition of μ, i.e. $\mu = \mu^+ - \mu^-$. Note that $F(Q)$ has the density $\sigma(\cdot, Q)$ with respect to Q and therefore

$$\frac{dF(Q)}{dP} = \sigma(\cdot, Q)\frac{dQ}{dP} \, .$$

With the use of Theorem 5, we obtain

$$\|F(Q) - F(R)\| = E^P \left| \frac{dF(Q)}{dP} - \frac{dF(R)}{dP} \right|$$

$$= \int_S \left| \sigma(x, Q)\frac{dQ}{dP}(x) - \sigma(x, R)\frac{dR}{dP}(x) \right| dP(x)$$

$$\leq \int |\sigma(x, Q) - \sigma(x, R)| \left| \frac{dQ}{dP}(x) \right| dP(x) + \int |\sigma(x, R)| \left| \frac{dQ}{dP}(x) - \frac{dR}{dP}(x) \right| dP(x)$$

$$\leq \int |\sigma(x, Q) - \sigma(x, R)| \, d|Q|(x) + \int |\sigma(x, R)| \left| \frac{dQ}{dP}(x) - \frac{dR}{dP}(x) \right| dP(x) \, .$$

Using the boundedness and Lipschitz conditions (16) and (15) in conjunction with Theorem 5, it follows that

$$\begin{aligned} \|F(Q) - F(R)\| &\leq L \|Q - R\| \, \|Q\| + \sigma_\infty \, \|Q - R\| \\ &\leq (2L + \sigma_\infty) \|Q - R\| \, . \end{aligned} \qquad (17)$$

Hence, F is Lipschitz continuous on the set of measures with variational norm less than 2.

To extend this property to $\tilde{F}$, we distinguish three cases. If both $\|Q\|, \|R\| \geq 2$, then $\tilde{F}(Q) = \tilde{F}(R) = 0$ and there is nothing to show. If $\|Q\| \geq 2 > \|R\|$, then

$$
\begin{aligned}
\left\| \tilde{F}(Q) - \tilde{F}(R) \right\| &= \left\| \tilde{F}(R) \right\| \\
&= (2 - \|R\|) \|F(R)\| \, .
\end{aligned}
$$

By Theorem 5 and the boundedness condition (16)

$$
\|F(R)\| = \int_S |\sigma(x, R)| \left| \frac{dR}{dP}(x) \right| dP(x) \leq \sigma_\infty \|R\| . \tag{18}
$$

Therefore,

$$
\begin{aligned}
\left\| \tilde{F}(Q) - \tilde{F}(R) \right\| &\leq (2 - \|R\|) \sigma_\infty \|R\| \\
&\leq 2(\|Q\| - \|R\|) \sigma_\infty \\
&\leq 2\sigma_\infty \|Q - R\| \, .
\end{aligned}
$$

If, finally, both $\|Q\|, \|R\| \leq 2$, then

$$
\begin{aligned}
\left\| \tilde{F}(Q) - \tilde{F}(R) \right\| &= \|(2 - \|Q\|)F(Q) - (2 - \|R\|)F(R)\| \\
&\leq (2 - \|Q\|) \|F(Q) - F(R)\| + \|F(R)\| \, |\|Q\| - \|R\|| \, .
\end{aligned}
$$

Now we use the Lipschitz continuity of F obtained in (17) and the upper bound for $\|F(R)\|$ of (18):

$$
\begin{aligned}
\left\| \tilde{F}(Q) - \tilde{F}(R) \right\| &\leq (2 - \|Q\|)(2L + \sigma_\infty) \|Q - R\| + 2\sigma_\infty \|Q - R\| \\
&\leq 4(L + \sigma_\infty) \|Q - R\| \, .
\end{aligned}
$$

This completes the proof of the Lemma. ∎

Lemma 4 *If the payoff function f is bounded, then conditions (15) and (16) of Lemma 3 are satisfied.*

Proof. We have to check that a bounded payoff function f implies the Lipschitz and boundedness conditions on σ assumed in Lemma 3. Let M be a bound for the payoff function f. Then

$$|E(Q,Q)| = \left| \int_S \int_S f(x,y)Q(dy)Q(dx) \right| \leq M\|Q\|^2$$

and

$$\left| \int_S f(x,y)Q(dy) \right| \leq M\|Q\|,$$

yield

$$|\sigma(x,Q)| \leq M\|Q\|(1 + \|Q\|) \tag{19}$$

whence (16) follows with $\sigma_\infty = 6M$.

For the Lipschitz condition (15), note that

$$
\begin{aligned}
|\sigma(x,Q) - \sigma(x,R)| &= \left| \int f(x,y)(Q-R)(dy) + E(R,R) - E(Q,Q) \right| \\
&\leq M\|Q-R\| + |E(R,R) - E(Q,Q)|.
\end{aligned}
$$

The bilinearity of E allows to write

$$|E(R,R) - E(Q,Q)| \leq |E(R,R-Q)| + |E(R-Q,Q)|.$$

But

$$
\begin{aligned}
|E(R,R-Q)| &= \left| \int_S \int_S f(x,y)R(dx)(R-Q)(dy) \right| \\
&\leq M\|R\|\|R-Q\|
\end{aligned}
$$

implies

$$|E(R,R) - E(Q,Q)| \leq M(\|R\| + \|Q\|)\|R-Q\|$$

and (15) follows. ∎

By the preceding lemmata in combination with Theorem 1 we know that the ordinary differential equation

$$Q'(t) = \tilde{F}(Q(t)), \; Q(0) = P \tag{20}$$

has a unique solution. For the replicator dynamics to be well defined we need furthermore that the set Δ of all populations is invariant under these dynamics. The proof of the following theorem appears in Bomze (1991, Lemma 2).

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Theorem 6 *Let $(Q(t))$ be the unique solution to (20) with initial condition $P \in \Delta$. Then $(Q(t)) \subset \Delta$, that is, the set of all populations Δ is invariant with respect to (20).*

Moreover, $Q(t)$ and P are uniformly equivalent in the sense that there exist constants $\gamma_t, \Gamma_t > 0$ such that

$$\gamma_t Q(t)(A) \leq P(A) \leq \Gamma_t Q(t)(A)$$

for all sets $A \in \mathcal{B}$. The density of $Q(t)$ with respect to P satisfies

$$\frac{dQ(t)}{dP}(x) = \exp\left(\int_0^t \sigma(x, Q(s))ds\right). \tag{21}$$

That is, if we start with a population $P \in \Delta$, the solution of $(Q(t))$ stays in the set of populations Δ. On Δ the functions $\tilde{F}$ and F coincide, which proves Theorem 2. The result on uniform equivalence is useful in Section 5.2.

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